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Local Dynamics of Symmetric Hamiltonian Systems with Application to the Affine Rigid Body

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To my parents

António and Esmeralda

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Declaration

This thesis is the original work of the author, with the exception of some sources cited in the text.

Maria Esmeralda Rodrigues de Sousa Dias
(February 1995)

Summary

This work is divided into two parts. The first one is directed towards the geometric theory of symmetric Hamiltonian systems and the second studies the so-called affine rigid body under the setting of the first part.

The geometric theory of symmetric Hamiltonian systems is based on Poisson and symplectic geometries. The symmetry leads to the conservation of certain quantities and to the reduction of these systems. We take special attention to the reduction at singular points of the momentum map. We survey the singular reduction procedures and we give a method of reducing a symmetric Hamiltonian system in a neighbourhood of a group orbit which is valid even when the momentum map is singular. This reduction process, which we called slice reduction, enables us to partially reduce the (local) dynamics to the dynamics of a system defined on a symplectic manifold which is the product of a symplectic vector space (symplectic slice) with a coadjoint orbit for the original symmetry group. The reduction represents the local dynamics as a coupling between vibrational motion on the vector space and generalized rigid body dynamics on the coadjoint group orbits. Some applications of the slice reduction are described, namely the application to the bifurcation of relative equilibria.

We lay the foundations for the study of the affine rigid body under geometric methods. The symmetries of this problem and their relationship with the physical quantities are studied. The symmetry for this problem is the semi-direct product of the cyclic group of order two $\mathbf{Z}_2$ by $SO(3) \times SO(3)$. A result of Dedekind on the existence of adjoint ellipsoids of a given ellipsoid of equilibrium follows as consequence of the $\mathbf{Z}_2$ symmetry. The momentum map for the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ action on the phase space corresponds to the conservation of the angular momentum and circulation. Using purely geometric arguments Riemann's theorem on the admissible equilibria ellipsoids for the affine rigid body is established. The symmetries of different relative equilibria are found, based on the study of the lattice of isotropy subgroups of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on the phase space. Slice reduction is applied in a neighbourhood of a spherical ellipsoid of equilibrium leading to different reduced dynamics. Based also on the slice reduction we establish the bifurcation of S-type ellipsoids from a nondegenerate ellipsoidal equilibrium.

Introduction

The modern geometric approach to Hamiltonian systems is based on the exploitation of the geometry of the phase space and the symmetries, which often are present in several areas of applied sciences. The symplectic and Poisson geometries offer a suitable framework for the Hamiltonian formulation of problems that appear in celestial mechanics, fluid and plasma dynamics and elasticity, among others. One of the main advantages of using this mathematical model for Hamiltonian systems is to enable to suppress the need of studying these systems in coordinate terms. This leads to a full generality of the theory and points to a qualitative study of the system.

In this work we treat symmetric Hamiltonian systems under this geometric setting taking special attention to the study of the symmetries. As it is well known the symmetries lead to conserved quantities which are expressed by the momentum map. The momentum map plays a key role on the reduction of these systems and in general, reduction consists in setting the momentum map equal to a constant, and quotient by a subgroup of the symmetry group which leaves invariant this level set.

The reduction for regular points of the momentum map is well understood (Meyer [35], Marsden and Weinstein [32]) and has many applications. Nevertheless, singular points of the momentum map are not an exception since they arise also as a consequence of the symmetry. Thus the extension of the theory to singular points is a need. The main difficulty with the singular reduction arises from the fact that the reduced spaces are not always manifolds, though they have a natural Poisson structure.

Here we survey the singular reduction and we give a partial reduction process for symmetric Hamiltonian systems in a neighbourhood of a group orbit, which works even for singular momentum mappings. This reduction, which we called slice reduction, factors out some symmetries to yield reduced systems that are defined on symplectic manifolds but which still have some symmetries left. The reduced symplectic manifolds are products of a symplectic vector space by a coadjoint group orbit of the original symmetry group and the reduction represents the dynamics as a coupling between vibrational motion on the vector space and generalized rigid body dynamics.

For application of this geometric approach to symmetric Hamiltonian systems, we chose the affine rigid body. The study of rotating fluid masses can be found among the works of several authors such as Newton, Maclaurin, Jacobi, Dedekind, Dirichlet, Riemann, Poincaré and others. Chandrasekhar [10] constitutes a comprehensive reference for these works.

The rich geometry of the phase space of the affine rigid body, the cotangent bundle of the group of 3×3 matrices with positive determinant, and the symmetry of this problem, fully justify its geometric treatment. The symmetry of the affine rigid body is the semi-direct product of the cyclic group of order two, $\mathbf{Z}_2$, by $SO(3) \times SO(3)$. Each factor of $SO(3) \times SO(3)$ is related to the rotation of the body and internal motions. As consequence of the $\mathbf{Z}_2$ symmetry we get a result of Dedekind on the existence of adjoint ellipsoids of a given ellipsoid of equilibrium. We give the physical interpretation of the geometric quantities and, in particular, it is shown that the components of the momentum map for the action of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on the phase space correspond to the conservation of the angular momentum and circulation.

The mathematical simplicity and the effectiveness of the referred geometric methods are evidenced by the results we achived without any study of the Hamiltonian. Among these we obtain Riemann's theorem on the admissible equilibria ellipsoids for the affine rigid body. The study of the lattices of isotropy subgroups of the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ action on the phase space and momentum space leads to the classification of the symmetries of different equilibria ellipsoids.

The slice reduction is applied in a neighbourhood of a spherical ellipsoid of equilibrium leading to different reduced dynamics. It is described also an application of the slice reduction to the bifurcation of relative equilibria. This result enables us to establish the bifurcation of S-type ellipsoids from a nondegenerate ellipsoidal equilibrium. For all these applications it is essential to know the slice representation of the isotropy subgroups of the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ action on the phase space. So, this representation has been determined for each isotropy subgroup of the lattice of the referred group action which enables also future applications.

Chapter 1

Reduction and Dynamics

Introduction

The modern treatment of Hamiltonian systems has as its main characteristic the exploration of the geometric properties not only of the phase space but in particular of the level sets of the so-called momentum map. In order to do so its study is done in the context of symplectic geometry or in a more general one the Poisson geometry. The symmetries often play an important role in the analysis of a particular system either as a symmetry of the system or a particular equilibrium, or as generator of a trajectory. The momentum map is the map which translates the conserved quantities due to the symmetries in the context of the symplectic and Poisson geometries.

We start this chapter by giving in section 1.1 some background information, namely basic definitions inherents to the context in which we will study symmetric Hamiltonian systems as well as some (classical) examples and results chosen either by necessity of its application in future sections or in order to point out some important properties.

Sections 1.2 and 1.3 review known results respectively for nonsingular and singular reductions of symmetric Hamiltonian systems. Section 1.2 also presents, as an example, the reduced Poisson system for the generalized rigid body problem which, under the mild hypothesis of the existence of an Ad-invariant inner product in the Lie algebra $\mathcal{G}$ (allowing the identification of $\mathcal{G}$ with $\mathcal{G}^*$), is given in terms of $\mathcal{G}^*$ coordinates by an Euler equation.

The main objective of section 1.3 is to give an account of the work that has been done over the last ten years on singular reduction, namely by doing a short comparision between very different kind of approaches.

The following section, section 1.4 can be viewed as a motivation and an introductory section of the next one. It gives the basic results necessary to obtain the reduction presented in section 1.5, which we called slice reduction and it works even for singular points of the momentum map. In particular section 1.4 explores the normal form of a momentum map giving the motivation and the conditions for its use.

Last section, 1.5, is constitued mainly by new results. It treats the reduction of symmetric Hamiltonian systems in a neighbourhood of a group orbit, $\mathcal{O}_p$, through a point p belonging to the phase space. It is shown that the dynamics in a neighbourhood of a group orbit can always be partially reduced to the dynamics of a system in symplectic slice coordinates and coadjoint group orbit coordinates. This system represents so the dynamics as a coupling of "vibrational" motions on the symplectic slice and generalized rigid body motions. In some particular cases this system represents also the full (reduced) dynamics.

1.1 Symplectic and Poisson Manifolds with a Group Action

The study of symmetric Hamiltonian systems will be done in the contexts of the Poisson or symplectic geometries and the symmetry is explored by using the so-called momentum map and its properties. Here we give a short account of the main set up in which (symmetric) Hamiltonian systems will be studied as well as some basic results. As main references for this section we cite Abraham and Marsden [1], Guillemin and Sternberg [20] and Marsden [31].

The phase space of a Hamiltonian system will be a Poisson or a symplectic manifold.

Definition 1 *Let $C^\infty(P)$ the set of the real-valued smooth functions on P. We say that $\{\ ,\ \}$ is a Poisson bracket if $\{\ ,\ \} : C^\infty(P) \times C^\infty(P) \to C^\infty(P)$, is bilinear, anticommutative, satisfies the Jacobi identity and the Leibniz's rule. A Poisson manifold is defined as a pair $(P, \{\ ,\ \})$ where $\{\ ,\ \}$ is a Poisson bracket.*

$\mathbf{R}^{2n+k}$ with coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n, t_1, \cdots, t_k)$ is a Poisson manifold with Poisson bracket given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

Note from this definition that the functions that are only functions of $(t_1, \cdots, t_k)$ are Casimirs, that is have zero bracket value with any other function.

Definition 2 *A symplectic manifold is a pair (P, ω) where ω is a closed nondegenerate differential two-form.*

$\mathbf{R}^{2n}$ with coordinates $(q_1, \cdots, q_n, p_1, \cdots, p_n)$ is a symplectic manifold with symplectic structure given by

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i, \tag{1.1}$$

which is called the canonical two-form.

The cotangent bundle T^*C of a given manifold C is another example of a symplectic manifold where the symplectic form is the canonical 2-form ω given in (1.1) being $q_i \in C$ and (q_i, p_i) the induced coordinates in T^*C.

Remark: Darboux's theorem (see Abraham and Marsden [1] pg.175) says that locally any symplectic structure of a $2n$-dimensional manifold is isomorphic to (1.1).

It is obvious from the above definitions that while a symplectic manifold is always even dimensional a Poisson manifold can be odd dimensional.

Examples of Poisson manifolds can be obtained from the next result which states that the dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ of a Lie group G is a Poisson manifold with a Poisson structure called Lie-Poisson structure.

Let $\langle\ ,\ \rangle$ be the pairing between $\mathcal{G}$ and $\mathcal{G}^*$.

Proposition 1 *$\mathcal{G}^*$, the dual of the Lie algebra $\mathcal{G}$ of a Lie group G, is a Poisson manifold with either of the two brackets*

$$\{f,g\}(\mu) = \pm\langle\mu,[Df_\mu, Dg_\mu]\rangle,$$

for $f,g : \mathcal{G}^* \to \mathbf{R}$, and $[\ ,\ ]$ the Lie algebra bracket.

Here Df_μ, Dg_μ are the derivatives of f,g regarded as functions in $\mathcal{G}$, i.e $Df_\mu, Dg_\mu \in (T_\mu\mathcal{G}^*)^* \cong \mathcal{G}$ where the isomorphism is given by:

$$\langle\nu, Df_\mu\rangle = \frac{d}{dt}f(\mu+t\nu)\Big|_{t=0} \qquad \nu\in\mathcal{G}^*, Df_\mu\in\mathcal{G}.$$

Note that in this proposition the antisymmetry of the Poisson bracket follows from the linearity of μ and the antisymmetry of the Lie bracket, the Jacobi identity follows from the Jacobi identity of the Lie bracket and from the fact:

$$\begin{aligned}\langle\nu, D\{g,h\}_\mu\rangle &= \frac{d}{dt}\{g,h\}(\mu+t\nu)\Big|_{t=0}\\ &= \frac{d}{dt}\langle\mu+t\nu,[Dg_{\mu+t\nu}, Dh_{\mu+t\nu}]\rangle\Big|_{t=0}\\ &= \langle\nu,[Dg_\mu, Dh_\mu]\rangle.\end{aligned}$$

So $D\{g,h\}_\mu = [Dg_\mu, Dh_\mu]$ and

$$\{f,\{g,h\}\}(\mu) = \pm\langle\mu,\left[Df_\mu, D\{g,h\}_\mu\right]\rangle = \pm\langle\mu,[Df_\mu,[Dg_\mu,Dh_\mu]]\rangle.$$

Leibniz's rule follows from the Leibniz's rule for the derivative of the product.

Hamiltonian vector fields can be defined on both Poisson and symplectic manifolds as follows:

Definition 3 *Let $(P,\{\ ,\ \})$ be a Poisson manifold (resp. (P,ω) a symplectic manifold) and $H\in C^\infty(P)$. The Hamiltonian vector field X_H is the unique vector field satisfying*

$$X_H(f) = \langle df, X_H\rangle = \{f,H\} \qquad \forall f\in C^\infty(P) \tag{1.2}$$

where $\langle,\rangle$ denotes the natural pairing between vectors and covectores, (respectively

$$\omega(X_H,Y) = dH\cdot Y).$$

Remark: The Poisson structure on a manifold P is a Lie algebra structure on $C^\infty(P)$ and $H\to X_H$ is a homomorphism from $C^\infty(P)$ to the set of vector fields in P, $\chi(P)$, as can easily be proved by the properties of the Poisson bracket. Also we have $X_{\{f,g\}} = [X_f, X_g]$.

Symplectic manifolds are particular cases of Poisson manifolds since a Poisson bracket of any two real valued smooth functions can be defined using a symplectic form by

$$\{f,g\} = \omega(X_f, X_g) \tag{1.3}$$

where X_f, X_g are Hamiltonian vector fields associated with f and g respectively.

Results relating Poisson and symplectic manifolds are not limited to the fact that symplectic manifolds are particular cases of Poisson manifolds. For instance Weinstein's result about the local structure of a Poisson manifold P known as the splitting theorem (see Weinstein [56]) states that there is a neighbourhood U of any point $p_0 \in P$ and an isomorphism ϕ from U to a product $S \times N$ of a symplectic manifold S by a Poisson manifold N such that the rank of N at $\phi(p_0)$ is zero. That is, locally P is the product of a symplectic by a Poisson manifold which are unique up to a local isomorphism. (See the referred work for the definition of rank of a Poisson structure at a point).

Another well known result is that every Poisson manifold is a union of symplectic leaves which in the case of $\mathcal{G}^*$ are the coadjoint group orbits.

Another important concept is the momentum map which plays a key role in the study of symmetric Hamiltonian systems namely in the reduction of a symmetric Hamiltonian system.

Let $\Phi : G \times P \to P$ be a Poisson (symplectic) action of the Lie group G on the Poisson (symplectic) manifold $(P, \{\ ,\ \})$ ((P,ω)), that is $\Phi_g : P \to P$, $p \mapsto \Phi_g(p) = \Phi(g,p)$ is a Poisson (symplectic) map.

Consider $\mathcal{G} = T_eG$ the Lie algebra of G, $\mathcal{G}^*$ its dual and $\langle , \rangle$ the pairing between $\mathcal{G}^*$ and $\mathcal{G}$ (that is $\langle \mu, \xi \rangle$, for $\xi \in \mathcal{G}, \mu \in \mathcal{G}^*$, denotes the value of the linear functional μ at ξ).

G acts on $\mathcal{G}$ and $\mathcal{G}^*$ respectively by the adjoint $Ad : G \times \mathcal{G} \to \mathcal{G}$, defined as $Ad_g = T_e(R_{g^{-1}} \circ L_g)$ where R and L denote respectively the right and left actions of G on itself, and co-adjoint $Ad^* : G \times \mathcal{G} \to \mathcal{G}^*$ actions ($\langle Ad_g^* \mu, \xi \rangle = \langle \mu, Ad_g \xi \rangle$).

(Symmetric) Hamiltonian systems can be defined on both Poisson and symplectic manifolds and the Hamiltonian flows ϕ_t will be respectively Poisson and symplectic maps for each t, that is maps which preserve the Poisson or symplectic structures in question.

Definition 4 *A Hamiltonian system is a triple $(P, \{\ ,\ \}, X_H)$ (resp. (P, ω, X_H)), where $(P, \{\ ,\ \})$ is a Poisson manifold (resp. (P,ω) is a symplectic manifold) and X_H is the Hamiltonian vector field associated with the Hamilton function $H : P \to \mathbf{R}$.*

A symmetric Hamiltonian system can be defined as a Hamiltonian system on $(P, \{\ ,\ \})$ ((P,ω)) where there is a Poisson (symplectic) group action Φ on P for which the Hamilton function H is invariant under Φ.

Note that due to the definition of Hamiltonian vector field the Hamilton's equations for $H \in C^\infty(P)$ can be written as:

1. $\dot{z} = X_H(z)$.

2. $\dot{f} = df(z) \cdot X_H(z) \qquad \forall f \in C^\infty(P)$.

3. $\dot{f} = \{f, H\} \qquad \forall f \in C^\infty(P)$.

Definition 5 *The infinitesimal generator corresponding to $\xi \in \mathcal{G}$ of the Φ action of G on P is*

$$\xi_P(p) = \frac{d}{dt}\Phi(\exp t\xi, p)\Big|_{t=0}.$$

Remark: The Tangent space to the group orbit $(G \cdot p)$ at p is given by

$$T_p(G \cdot p) = \{\xi_P(p) | \xi \in \mathcal{G}\}.$$

Definition 6 (Momentum map) *We say that $J : P \to \mathcal{G}^*$ is a momentum map for the G-action on P provided that for all $\xi \in \mathcal{G}$*

$$X_{\widehat{J}(\xi)} = \xi_P,$$

where $\widehat{J}(\xi) : P \to \mathbf{R}$ is $\widehat{J}(\xi)(p) = \langle J(p), \xi \rangle$.

Remark: For the symplectic case the definition of the momentum map given above is equivalent to the following condition

$$\langle DJ_p \cdot v, \xi \rangle = \omega(\xi_P(p), v) \qquad \forall v \in T_pP \tag{1.4}$$

Property: If H is a G-invariant Hamilton function then J is an integral for X_H, i.e

$$dH \,.\, \xi_P(p) = 0$$

$$\widehat{J}(\xi)(F_t(p)) = \widehat{J}(\xi)(p)$$

with F_t the flow of X_H.(See Abraham and Marsden [1] pg. 277).

We can define, though does not always exists, an equivariant momentum map as a momentum map J for which the following diagram commutes:

$$\begin{tikzcd}
P \arrow[r, "J"] \arrow[d, dash, "\Phi_g"'] & \mathcal{G}^* \arrow[d, dash, "Ad^*_{g^{-1}}"] \\
P \arrow[r, "J"'] & \mathcal{G}^*
\end{tikzcd}$$

Almost for all cases we meet in practice we have an equivariant momentum map. However let us to specify two of them for which a symplectic group action of G on a symplectic manifold (P, ω) have an equivariant momentum map:

a) P is a cotangent bundle.

b) G is a compact Lie group.

The equivariance for **a)** is an imediate consequence of the canonical two-form be an exact form. Note also that in this case we not only get the equivariance of the momentum map but as well a particular expression for this map.

A cotangent bundle $P = T^*Q$ is a symplectic manifold, so a Poisson manifold, with the canonical two-form ω_0. The form ω_0 is an exact form, i.e $\omega_0 = -d\theta_0$ where θ_0 is the canonical one-form, defined by

$$\langle \theta_0(\alpha_q), \omega_{\alpha_q} \rangle = \langle T\tau_Q^* \omega_{\alpha_q}, \alpha_q \rangle$$

with $\omega_{\alpha_q} \in T_{\alpha_q}(T^*Q)$, $\alpha_q \in T_q^*Q$ and $\tau_Q^* : T^*Q \to Q$ the cotangent bundle projection. The fact of ω_0 be an exact two-form implies equivariance of the momentum map for a group action on $P = T^*Q$, induced from the one on Q, called the lifted action Φ^{T^*} of the Φ action on Q. Φ^{T^*} is defined via the lift $T^*\Phi_g$ of Φ_g as $\Phi_g^{T^*} = T^*\Phi_{g^{-1}}$ where

$$\begin{gathered} T^*\Phi_g : T^*Q \to T^*Q \\ \langle T^*\Phi_g(\alpha_q), v \rangle = \langle \alpha_q, T\Phi_g \cdot v \rangle \end{gathered} \tag{1.5}$$

whith $v \in T_{\Phi_{g^{-1}}(q)}Q$ and $T\Phi_g$ the tangent map.

Proposition 2 *Let Φ be the action of G on Q and $\Phi^{T^*}(g, \alpha) = T^*\Phi_{g^{-1}}(\alpha)$ the G action on $P = T^*Q$. The momentum map for the G action on $P = T^*Q$ is the Ad^*-equivariant momentum map $J : T^*Q \to \mathcal{G}^*$ given by*

$$\langle J(\alpha_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle \tag{1.6}$$

where ξ_Q is the infinitesimal generator associated to ξ for the Φ action.

(See corollary 4.2.11 of Abraham and Marsden [1] for this result.)

For compact group actions on a symplectic manifold the result of the equivariance of the momentum map involves some results which we will try to summarize here.

From theorem 26.1 of Guillemin and Sternberg [20] the condition:

$$H^1(\mathcal{G}) = H^2(\mathcal{G}) = \{0\} \tag{1.7}$$

where $H^1(\mathcal{G})$ and $H^2(\mathcal{G})$ denote respectively the 1st and 2nd cohomology of the Lie algebra $\mathcal{G}$ of G, guarantees the existence of an equivariant momentum map for the symplectic action of G on (P, ω). (See Guillemin and Seternberg [20], pg.170, for the definition of the kth cohomology of a Lie algebra $\mathcal{G}$, $H^k(\mathcal{G})$).

The condition (1.7) is verified when the Lie algebra $\mathcal{G}$ is semisimple. The semisimplicity of a Lie algebra is given by Cartan's criterion in terms of the nondegeneracy of the so-called Killing form of a Lie algebra (see Sattinger and Weaver [42] and Guillemin and Sternberg [20] §52 for Cartan's criterion) . That is, a Lie algebra is semisimple if and only if its Killing form $K(X, Y)$ is nondegenerate, where

$$K(X, Y) = \text{tr}\ (ad_X ad_Y)$$

with $ad_X : Z \mapsto [X, Z]$ the adjoint representation of $\mathcal{G}$ and tr denoting the "trace".

The compactness of a (real) Lie group G is equivalent to the negative definitess of the Killing form (see Weyl's theorem of Sattinger and Weaver [42]). So if G is compact its Lie algebra is semisimple and then condition (1.7) is verified guaranting the equivariance of the momentum map.

From the definition of momentum map and as a consequence of the equivariance, we have on a symplectic manifold the following (infinitesimal) equality

$$\begin{aligned} \omega\left(X_{\mathcal{J}(\xi)}(x), X_{\mathcal{J}(\eta)}(x)\right) &= \omega\left(\xi_P(x), \eta_P(x)\right) = -\langle DJ_x\left(\xi_P(x)\right), \eta\rangle \\ &= \langle J(x), [\xi, \eta]\rangle. \end{aligned} \tag{1.8}$$

This equality allows to show that any equivariant momentum map is a Poisson map, that is a smooth map $J : P_1 \to P_2$ between two Poisson manifolds P_1 and P_2 , such that for all $f, g \in C^\infty(P_2)$ the following condition holds

$$\{f \circ J, g \circ J\}_1 = \{f, g\}_2 \circ J,$$

where $\{\ ,\ \}_1, \{\ ,\ \}_2$ are the Poisson structures of P_1 and P_2 respectively.

Proposition 3 *If $J : P \to \mathcal{G}^*$ is an equivariant momentum map then is a Poisson map.*

Proof: We need to prove that for all $f, g : \mathcal{G}^* \to \mathbf{R}$

$$\{f, g\}_{\mathcal{G}^*}(J(x)) = \{f \circ J, g \circ J\}_P(x).$$

By proposition 1

$$\begin{aligned} \{f, g\}_{\mathcal{G}^*}(J(x)) &= \pm\langle J(x), \left[Df_{J(x)}, Dg_{J(x)}\right]\rangle \\ &= \pm\omega\left(X_{\mathcal{J}(Df_{J(x)})}(x), Y_{\hat{\mathcal{J}}(Dg_{J(x)})}(x)\right) && \text{by (1.8)} \\ &= \pm\left\{\hat{\mathcal{J}}\left(Df_{J(x)}\right), \hat{\mathcal{J}}\left(Dg_{J(x)}\right)\right\}(x) && \text{by (1.3)} \end{aligned}$$

where $\hat{\mathcal{J}}\left(Df_{J(x)}\right) : x \mapsto \langle J(x), Df_{J(x)}\rangle$. From the identification of $Df_{J(x)}$ with an element of $\mathcal{G}$ we have:

$$\begin{aligned} \langle J(x), Df_{J(x)}\rangle &= \frac{d}{dt} f\left(J(x) + tJ(x)\right)\Big|_{t=0} \\ &= \frac{d}{dt} f\left(J(x)\right) + tf\left(J(x)\right)\Big|_{t=0} && \text{by linearity of f} \\ &= (f \circ J)(x). \end{aligned}$$

The result now follows.

□

Conserved quantities are a consequence of the presence of the symmetry and expressed by the momentum map. This is illustrated by the conservation of the angular momentum in the case of $G = SO(3)$ and $T^*\mathbf{R}^3$.

Example 1 (*Conservation of the angular momentum*):

Consider the left action of $SO(3)$ on $\mathbf{R}^3$:

$$\begin{aligned} L \;\; : SO(3) \times \mathbf{R}^3 &\longrightarrow \mathbf{R}^3 \\ (\Lambda, x) &\longmapsto \Lambda\, x. \end{aligned}$$

We can identify the Lie algebra of $SO(3)$, $so(3)$, with $\mathbf{R}^3$ using the following isomorphism $\mathbf{i} : \mathbf{R}^3 \longrightarrow so(3)$

$$x = (x_1, x_2, x_3) \;\; \longmapsto \;\; \hat{x} = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \in so(3). \tag{1.9}$$

Indeed $\mathbf{i}$ defines an isomorphism of the Lie algebras $(\mathbf{R}^3, \times)$ and $(so(3), [\, , \,])$, where $\times$ is the vector product in $\mathbf{R}^3$ and $[\, , \,]$ the Lie bracket in $so(3)$, since $\widehat{x \times y} = \hat{x}\hat{y} - \hat{y}\hat{x}$.

Defining the pairing between $\mathbf{R}^{3^*}$ and $\mathbf{R}^3$ as the inner product of two vectors we can also identify $so(3)^*$ with $\mathbf{R}^3$ since $x \cdot y = \frac{1}{2} tr(\hat{x}^T \hat{y})$.

The infinitesimal generator for the L-action corresponding to $\hat{\xi} \in so(3)$ is given by

$$\hat{\xi}_{\mathbf{R}^3}(x) = \frac{d}{dt}\Phi(\exp t\hat{\xi}, x)\Big|_{t=0} = \hat{\xi}x = \xi \times x.$$

So for $\alpha_x \in T^*_x\mathbf{R}^3$ the momentum map $J : T^*\mathbf{R}^3 \longrightarrow so(3)^*$, for the lift action of Φ to $T^*\mathbf{R}^3 = \mathbf{R}^3$, is given by

$$\begin{aligned} \langle J(\alpha_x), \hat{\xi}\rangle &= \langle \alpha_x, \hat{\xi}_{\mathbf{R}^3}(x)\rangle &&= \langle \alpha_x, \xi \times x\rangle \\ &= \alpha_x \cdot (\xi \times x) &&= \xi \cdot (x \times \alpha_x) \\ &= \langle x \times \alpha_x, \hat{\xi}\rangle &&= \frac{1}{2}\langle [\hat{x}, \widehat{\alpha_x}], \hat{\xi}\rangle. \end{aligned}$$

That is $J(\alpha_x) = [\hat{x}, \widehat{\alpha_x}] = x \times \alpha_x$, which is the *angular momentum.*

It is worth to include here some aspects of the so-called generalized rigid body problem namely the so-called body and space coordinates of phase space. These coordinates play a key role in the normal form for the momentum map which will be studied in future sections.

Example 2 (*Momentum map for the generalized rigid body*):

A rigid body can be defined as a set of particles such that there is a frame relative to which all the particles are at rest at all the times. Clearly the study of the rigid body motion involves two kinds of frames F and $\hat{F}$, one is fixed outside the body and the other moving with the body. As a frame is a set of a origin and a orthonormal basis, the transition from one frame to another is given by multiplication of a orthogonal matrix. The condition of the existence of one frame relative to which the distance between all the particles are kept fixed allows to consider the configuration space for the rigid body as $O(3)$ or $SO(3)$ if an orientation is also given. Obviously one can express the body and its motion in two types of coordinates, either relative to the orthonormal basis in the body (body coordinates) or relative to the orthonormal basis in the space (space coordinates). The transition from one to another is by multiplication on the left or on the right by a orthogonal matrix.

An element Λ of $SO(3)$ giving a configuration of a body is regarded as a map of a reference configuration $\mathcal{B} \subset \mathbf{R}^3$ to the current configuration $\Lambda(\mathcal{B})$, i.e

$$X \mapsto \Lambda X = x.$$

For the rigid body, Λ becomes time dependent and the velocity of a point of the body is $\dot{x} = \dot{\Lambda} X = \dot{\Lambda}\Lambda^{-1} x = \omega \times x$. ω defines the spatial angular velocity and $\Omega = \Lambda^{-1}\omega$ the body angular velocity. The phase space is taken to be the cotangent space $T^*SO(3)$.

A generalized rigid body is one where the configuration space is taken to be a general Lie group G and the phase space its cotangent bundle.

Consider G acting on T^*G via the lift action of $\Phi : G \times G \to G$ where

$$\Phi(g,h) = gh = L(g,h)$$

is the left action of G on G, which we denote here by Φ to avoid any danger of confusion. The momentum map for the lifted action is given by

$$\langle J(\alpha_g), \xi \rangle = \langle \alpha_g, \xi_G(g) \rangle = \langle \alpha_g, \frac{d}{dt} \exp t\xi \cdot g\Big|_{t=0} \rangle = \langle \alpha_g, T_e R_g(\xi) \rangle \tag{1.10}$$

where $\xi \in T_e G = \mathcal{G}$ and R is the right action of G on G given by

$$\begin{gathered} R : G \times G \to G \\ R(g,h) = hg. \end{gathered}$$

So the momentum map (1.10) is just

$$\langle J(\alpha_g), \xi \rangle = \langle \alpha, T_e R_g(\xi) \rangle = \langle (T_e R_g)^* \alpha, \xi \rangle \qquad \alpha \in T_g^* G \tag{1.11}$$

where $T_e R_g$ denotes the tangent at the identity and $(T_e R_g)^*$ its dual map. That is $J(\alpha_g) = (T_e R_g)^* \alpha$.

We can identify T^*G with $G \times \mathcal{G}^*$ using two isomorphisms $\mathbf{i}_B, \mathbf{i}_S$ such that the image by $\mathbf{i}_B$ ($\mathbf{i}_S$) of an element of T^*G represents this element in the so-called *body* (*space*) coordinates.

First of all note that $\alpha \in T^*G$ is a 1-form, i.e for $\tau_G^*(\alpha) = g$, $\alpha : T_g G \to \mathbf{R}$, $u \mapsto \langle \alpha, u \rangle$. The left and right actions induce maps on the tangent space, respectively L_{g^*} and R_{g^*} given by

$$\begin{aligned} L_{g^*} &: T_e G \to T_g G & R_{g^*} &: T_e G \to T_g G \\ L_{g^*} &= T_e L_g & R_{g^*} &= T_e R_g \end{aligned}$$

If $\tau_G^*(\alpha) = g$, then $\langle \alpha, u \rangle = \langle \alpha, T_e L_g(\xi) \rangle \stackrel{\text{def}}{=} \langle (T_e L_g)^* \alpha, \xi \rangle$ for the left action, or $\langle \alpha, u \rangle = \langle \alpha, T_e R_g(\xi) \rangle = \langle (T_e R_g)^* \alpha, \xi \rangle$ for the right action. That is $(T_e L_g)^* \alpha$ and $(T_e R_g)^* \alpha$ are elements of $\mathcal{G}^*$ and so the following isomorphisms:

$$\begin{aligned} \mathbf{i}_B &: T^*G \to G \times \mathcal{G}^* & \mathbf{i}_S &: T^*G \to G \times \mathcal{G}^* \\ \mathbf{i}_B(\alpha) &= (g, (T_e L_g)^* \alpha) & \mathbf{i}_S(\alpha) &= (g, (T_e R_g)^* \alpha). \end{aligned} \tag{1.12}$$

where $\tau_G^*(\alpha) = g$.

The momentum map for the lifted action in body and space coordinates respectively J_B, J_S is given by

$$\begin{aligned} J_B(h,\mu) = J \circ \mathbf{i}_B^{-1}(h,\mu) &= J((T_eL_{h^{-1}})^*\mu) = (T_eR_h)^* \circ (T_eL_{h^{-1}})^*\mu \\ &= (T_e(L_{h^{-1}} \circ R_h))^*\mu = Ad^*_{h^{-1}}\mu \end{aligned} \tag{1.13}$$

$$\begin{aligned} J_S(h,\mu) = \left(J \circ \mathbf{i}_S^{-1}\right)(h,\mu) &= J\left((T_eR_{h^{-1}})^*\mu\right) \\ &= (T_eR_h)^* \circ (T_eR_{h^{-1}})^*\mu = (T_e(R_e))^*\mu = \mu. \end{aligned} \tag{1.14}$$

We can also compute the expressions of the lifted action Φ^{T^*} in body and space coordinates either by using its definition or the equivariance of the momentum map. These expressions (see Abraham and Marsden [1] pg. 317) are:

i) In body coordinates: $\Phi_B^{T^*}(g,(h,\mu)) = (gh,\mu)$.

ii) In space coordinates: $\Phi_S^{T^*}(g,(h,\mu)) = \left(gh, Ad^*_{g^{-1}}(\mu)\right)$.

□

1.2 Nonsingular Reduction

Since the early seventies with the works of Meyer [35] and Marsden and Weinstein [32] the reduction of symmetric Hamiltonian systems has interested several authors whose have developed generalizations and applications of these results. Roughly speaking the reduction of symmetric Hamiltonian systems consists in reducing the study of a given system to one in a smaller phase space in such a way that we are able to deduce results about the original system from the reduced system.

The reduction in the nonsingular cases is very well studied and one finds terms such as Poisson reduction, symplectic reduction, cotangent bundle reduction, for the reduction of a symmetric Hamiltonian system defined on a phase space which is a Poisson, symplectic or cotangent manifold respectively. All these results can be found in Marsden [31] which we will follow here as main reference.

In this section we survey some reduction procedures which result in nice reduced spaces, such as the ones which are manifolds, and at the same time we explore some examples of reduced spaces and their properties. In particular we deduce the reduced Hamiltonian system of a given system in T^*G in terms of $\mathcal{G}^*$ coordinates which we shall call coadjoint rigid body coordinates.

The reduction procedures given here are a consequence of the symmetry of the system and, as we will see below, the reduced phase space is the quotient of a set $S \subseteq P$ by a subgroup $T \subseteq G$, where P is the phase space and G is the Lie group acting on it.

Let P be a Poisson (resp. symplectic) manifold and $\Phi : G \times P \to P$ the action of the Lie group G on P by Poisson (resp. symplectic) maps.

Reduced Poisson Manifold:

Let $(P, \{\ ,\ \})$ be a Poisson manifold and P/G a smooth manifold. We define the *reduced Poisson manifold* as the Poisson manifold $(P/G, \{\ ,\ \}_{P/G})$, where $\{\ ,\ \}_{P/G}$ is the Poisson structure induced from the Poisson structure $\{,\}_P$ on P and given by

$$\{f, g\}_{P/G}(\pi(p)) = \{f \circ \pi, g \circ \pi\}_P(p)$$

for f, g smooth real-valued functions defined on P and $\pi : P \longrightarrow P/G$ the canonical projection.

Note that the functions $f \circ \pi, g \circ \pi$ are G-invariants since the action of G on P is by Poisson maps, so $\{f \circ \pi, g \circ \pi\}_P$ is G-invariant and $\{\ ,\ \}_{P/G}$ is well defined. Thus $\left(C^\infty(P/G), \{\ ,\ \}_{P/G}\right)$ is a Poisson algebra.

A result of Schwarz [43] gives that the Lie algebra $C^\infty(P/G)$ isomorphic to the space of smooth G-invariant functions $C^\infty(P)^G$.

Example 3 (The reduced Poisson manifold for $P = T^*G$)

By example 2 we have that the momentum map for the action of G on T^*G by cotangent lift of the left and right translations is respectively:

$$J_L(\alpha_g) = (T_eR_g)^*\,\alpha_g$$

and

$$J_R(\alpha_g) = (T_eL_g)^*\,\alpha_g,$$

where $\alpha_g \in T^*G$.

By proposition 3 these maps are Poisson maps if we take for the Lie-Poisson structure in $\mathcal{G}^*$ the minus in the case of J_R and the plus for J_L.

So $\mathcal{G}^*$ with the minus (plus) Lie Poisson bracket is the reduced Poisson manifold (T^*G/G) for the cotangent lift of the left (right) action of G on G, being J_R (J_L) the quotient map.

Note also that if we consider the Lie algebra $(C^\infty(\mathcal{G}^*), \{\ ,\ \}_\pm)$ with the Lie-Poisson structure and the coadjoint action of G on $\mathcal{G}^*$ then for all $\phi, \psi \in C^\infty(\mathcal{G}^*)$ with ϕ a G-invariant function, i.e

$$\phi(Ad^*_{g^{-1}}\mu) = \phi(\mu) \qquad \forall \mu \in \mathcal{G}^*,\ \forall g \in G \tag{1.15}$$

we have

$$\{\phi, \psi\}(\mu) = 0 \qquad \forall \mu \in \mathcal{G}^*.$$

This can easily be obtained by taking in (1.15) $g = \exp -t\xi$ and differentiating it with respect to t at $t = 0$ and making use of the the identification of $D\phi_\mu, D\psi_\mu$ as elements of $\mathcal{G}$ (see proposition 1). That is

$$\langle \mu, [\xi, D\phi_\mu] \rangle = 0 \qquad \forall \xi \in \mathcal{G}, \forall \mu \in \mathcal{G}^*.$$

For $\xi = D\psi_\mu$ the above expression is equivalent to

$$\{\phi, \psi\}(\mu) = \pm \langle \mu, [D\phi_\mu, D\psi_\mu] \rangle = 0$$

This can be rephrased by saying that the Poisson structure in $\mathcal{G}^*$ is a union of (symplectic) leaves, the coadjoint orbits.

□

Reduced Symplectic Manifold:

Let (P,ω) be a symplectic manifold, J an Ad^* equivariant momentum map for the symplectic action of the Lie group G on P and G_μ the isotropy subgroup of $\mu \in \mathcal{G}^*$ for the coadjoint action, that is

$$G_\mu = \left\{ g \in G : Ad^*_{g^{-1}}\mu = \mu \right\}.$$

Suppose that $J^{-1}(\mu)/G_\mu$ is a smooth manifold. We define the *reduced symplectic manifold* as (P_μ, ω_μ) where $P_\mu = J^{-1}(\mu)/G_\mu$ and ω_μ is the unique symplectic form induced from the symplectic form ω on P given by

$$i^*_\mu \omega = \pi^*_\mu \omega_\mu \qquad \Longleftrightarrow \qquad \omega_\mu([v],[u]) = \omega(v,u)$$

for $v,u \in T_p J^{-1}(\mu)$ with $p \in J^{-1}(\mu)$, $[v],[u] \in J^{-1}(\mu)/G_\mu$, i^*_μ the pull-back of the inclusion $i_\mu : J^{-1}(\mu) \to P$ and π^*_μ the pull-back of the canonical projection π_μ : $J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$, that is

$$P \overset{i_\mu}{\leftarrow} J^{-1}(\mu) \overset{\pi_\mu}{\to} J^{-1}(\mu)/G_\mu.$$

Remark : Two conditions are usually assumed in order to guarantee that $J^{-1}(\mu)/G_\mu$ is a manifold:

1. μ is a regular point of J which implies that $J^{-1}(\mu)$ is a manifold.
2. The action of G_μ on $J^{-1}(\mu)$ is free and proper which implies that $J^{-1}(\mu)/G_\mu$ is a manifold.

Example 4 (The reduced symplectic manifold for $P = T^*G$)

Let the G-action on $P = T^*G$ be the lift of the left action of G on G. The momentum map is $J(\alpha_g) = (T_e R_g)^* \alpha_g$ (see example 2) and

$$J^{-1}(\mu) = \{\alpha_g \in T^*G : \langle \alpha_g, \xi_G(g)\rangle = \langle \mu, \xi\rangle\},$$

for $\mu \in (T_e G)^* = \mathcal{G}^*$.

The infinitesimal generator for the (left) G-action on G, $\xi_G(g)$, is a right invariant vector field, that is

$$T_h R_g\left(\xi_G(h)\right) = \xi_G\left(R_g(h)\right) \tag{1.16}$$

as easily follows from the expression of ξ_G. Thus $\xi = \xi_G(e) = T_g R_{g^{-1}}\left(\xi_G(g)\right)$. Hence the condition $\langle \alpha_g, \xi_G(g)\rangle = \langle \mu, \xi\rangle$ is equivalent to:

$$\langle \alpha_g, \xi_G(g)\rangle = \langle \mu, T_g R_{g^{-1}} \xi_G(g)\rangle = \langle (T_g R_{g^{-1}})^* \mu, \xi_G(g)\rangle.$$

This gives $J^{-1}(\mu)$ as the graph of a one form, say $\alpha_\mu(g) = (T_g R_{g^{-1}})^* \mu$, i.e

$$J^{-1}(\mu) = \{\alpha_g \in T^*G : (T_g R_{g^{-1}})^* \mu = \alpha_g\}$$

The action of G_μ on $J^{-1}(\mu)$ is the restriction to G_μ of the G action on T^*G, that is $T^*L_{g^{-1}} = L_g^*$, and it acts on the 1-form α_μ by translation of the base point. Indeed, by definition of lift:

$$\begin{aligned}\langle T^*L_{g^{-1}}\alpha_\mu(h), \xi_G(gh)\rangle &= \langle \alpha_\mu(h), T_{gh}L_{g^{-1}}\xi_G(gh)\rangle = \langle \mu, T_hR_{h^{-1}}[g^{-1}\xi gh]\rangle \\ &= \langle Ad_g^*\mu, \xi\rangle = \langle \mu, \xi\rangle = \langle \alpha_\mu(gh), \xi_G(gh)\rangle.\end{aligned}$$

That is $T^*L_{g^{-1}}\alpha_\mu(h) = \alpha_\mu(gh)$. So

$$G_\mu = \left\{ g \in G : L_g^*\alpha_\mu = \alpha_\mu \right\}.$$

This means that $J^{-1}(\mu)/G_\mu \cong G/G_\mu$, and G/G_μ is diffeomorphic to $G\cdot\mu$ (the coadjoint group orbit through μ), with the diffeomorphism given by $\phi : J^{-1}(\mu)/G_\mu \to G\cdot\mu$ where $\phi : \pi_\mu(\alpha_\mu(g)) \mapsto Ad_{g^{-1}}^*\mu$ with π_μ the canonical projection. That is, the reduced phase space is isomorphic to the coadjoint orbit through μ,

$$(T^*G)_\mu = \mathcal{O}_\mu \qquad \mathcal{O}_\mu = \left\{ Ad_{g^{-1}}^*\mu \right\}.$$

□

Comparing the reduced Poisson manifold for $P = T^*G$ and the reduced symplectic manifold, obtained in examples 3 and 4, i.e respectively $\mathcal{G}^*$ and $\mathcal{O}_\mu$ we can say that the Lie Poisson structure on $\mathcal{G}^*$ is compatible with the coadjoint orbit symplectic struture, known as Kostant-Kirillov-Souriau, and given by:

$$\omega_\mu(\nu)\left(\xi_{\mathcal{G}^*}(\nu), \eta_{\mathcal{G}^*}(\nu)\right) = -\langle \nu, [\xi, \eta]\rangle \tag{1.17}$$

for $\mathcal{G}^* \ni \nu = Ad_{g^{-1}}^*\mu$ and $\xi_{\mathcal{G}^*}$ the infinitesimal generator for the coadjoint action on $\mathcal{G}^*$ (see Abraham and Marsden [1], pg.303, for the coadjoint orbit symplectic form).

To say that the Poisson and symplectic structures are compatible means that

$$\{f, g\}_{\mathcal{O}_\mu} = \left\{\overline{f}, \overline{g}\right\}_{\mathcal{G}^*}\Big|_{\mathcal{O}_\mu}$$

where $\overline{f}, \overline{g}$ are arbitrary extensions of f, g to $\mathcal{G}^*$, and $\mathcal{O}_\mu$ is the coadjoint orbit through μ.

Remark: There is a more general relation between the Poisson and symplectic reduced spaces relating the foliation of a Poisson manifold into symplectic leaves to the reduced symplectic manifolds.

Away from singular points the reduced symplectic manifold $J^{-1}(\mu)/G_\mu$ is symplectic isomorphic to $P_{\mathcal{O}_\mu} = J^{-1}(\mathcal{O}_\mu)/G$ where $\mathcal{O}_\mu$ is the coadjoint group orbit through μ.

From the equivariance of the momentum map we have that this isomorphism is induced from the inclusion $i : J^{-1}(\mu) \to J^{-1}(\mathcal{O}_\mu)$ by taking equivalence classes. The symplectic structure on $P_{\mathcal{O}_\mu}$ is induced from the given symplectic form ω on P and from the orbit symplectic form $\omega_{\mathcal{O}_\mu}$ (see Marsden [31], pg. 38).

Cotangent Bundle Reduction

When the phase space P is a cotangent bundle, say $P = T^*Q$, and we are away from singular points of the momentum map, there are two ways of interpreting the reduced phase space summarized in the following diagram.

$$\begin{tikzcd}
T^*Q \arrow[r, phantom, "\supset"] \arrow[ddd] & J^{-1}(\mu) \arrow[r, phantom, "\subset"] \arrow[d] & J^{-1}(\mathcal{O}_\mu) \arrow[r, phantom, "\subset"] \arrow[d] & T^*Q \arrow[ddd] \\
& P_\mu \arrow[r, phantom, "\cong"] \arrow[d, "\text{injection}"'] & P_{\mathcal{O}} \arrow[d, "\mathcal{O}"', "\text{surjection}"] & \\
& T^*(Q/G_\mu) \arrow[d] & T^*(Q/G) \arrow[d] & \\
Q \arrow[r] & Q/G_\mu \arrow[r] & Q/G & Q \arrow[l]
\end{tikzcd}$$

One version represented by the injection in the diagram above sees the reduced space P_μ as an embedded vector subbundle of $T^*(Q/G_\mu)$ (see Abraham and Marsden [1]). The other version states that the reduced space $J^{-1}(\mu)/G_\mu$ is a coadjoint bundle over $T^*(Q/G)$ with fiber the coadjoint orbit through μ, $\mathcal{O}_\mu$, and is represented in the diagram by the surjection (see Marsden [31], Lewis, Marsden, Ratiu and Simo [25]).

Both versions are proved by making use of the properties of the so-called mechanical connection, which is a G-equivariant map α:

$$\alpha : TQ \to \mathcal{G}$$

such that

$$\alpha(\xi_Q(q)) = \xi \qquad \text{for all} \quad \xi \in \mathcal{G}.$$

A mechanical connection enables us to define a one-form α_μ with values in $J^{-1}(\mu)$ as follows

$$\langle \alpha_\mu(q), v\rangle = \langle \mu, \alpha(q,v)\rangle \qquad (q,v) \in TQ.$$

The shifting map Σ, where $\Sigma(z) = z - \alpha_{J(z)}$ for $z \in T^*_qQ$, relates the reduction at zero with the orbit reduction, since Σ is an equivariant map and $\Sigma : J^{-1}(\mathcal{O}_\mu) \to J^{-1}(0)$.

The reduction at zero is simpler, the reduced space $P_0 = (T^*Q)_0 = J^{-1}(0)/G$ can be identified with $T^*(Q/G)$ since for $\beta_q \in J^{-1}(0)$ we have

$$\langle \beta_q, \xi_Q(q)\rangle = 0 \qquad \text{for all} \quad \xi \in \mathcal{G}.$$

That is β_q can be viewed as a one-form on $T^*(Q/G)$.

The equivariance of the shifting map now gives:

$$(T^*Q)_{\mathcal{O}_\mu} = J^{-1}(\mathcal{O}_\mu)/G \to J^{-1}(0)/G = T^*(Q/G).$$

Furthermore as $\Sigma \circ \alpha_\mu = 0$ for all $\nu \in \mathcal{O}_\mu$, the fiber of Σ is $\mathcal{O}_\mu$ and the reduced space is a coadjoint bundle with fiber the coadjoint orbit.

The symplectic and Poisson reduced structures are induced from the symplectic form on $T^*(Q/G_\mu)$. For details see the cited references.

Reduced Hamiltonian System

Both the Poisson and symplectic reduction produce a reduced phase space which is a quotient, say $R = S/T$, of a manifold $S \subseteq P$ by a subgroup T of G. For a G-invariant Hamiltonian function the *reduced Hamiltonian function* H_R is given by

$$H_R \circ \pi = H \circ i \tag{1.18}$$

where $\pi : S \to S/T$ is the canonical projection and $i : S \to P$ is the inclusion.

The flow F_t of the Hamiltonian vector field X_H induces a H-equivariant flow ϕ_t on the reduced space with $\pi \circ F_t = \phi_t \circ \pi$.

In the Poisson case this follows from the definition of the bracket of the reduced space $\{\ ,\ \}_{P/G}$ and of Hamiltonian vector fields on Poisson manifolds. In particular given a Hamiltonian $H_R \in C^\infty(P/G)$ an integral curve of H_R through $m_0 \in P/G$ is a curve $\gamma_t(m_0) = \gamma(t)$ with $\gamma(0) = m_0$ such that for all functions $f \in C^\infty(P/G)$

$$\frac{d}{dt} f(\gamma(t)) = \{f, H_R\}_{P/G}(\gamma(t))$$

For the symplectic case it must also be taken in account that $J^{-1}(\mu)$ is G_μ-invariant manifold. The flow is Hamiltonian on P_μ with Hamiltonian function H_μ given by the expression (1.18) and one can deduce the flow of the original system from the flow of the reduced system by the process given in (Abraham and Marsden [1] pg. 305).

Example 5 (Generalized rigid body dynamics)

The reduced system for the generalized rigid body can be given in $\mathcal{G}^*$ coordinates if one can identify $\mathcal{G}$ with $\mathcal{G}^*$, say via an inner product. This is the case when G is compact.

Here we deduce the reduced (Poisson) Hamiltonian system in $\mathcal{G}^*$ coordinates. This reduced system when restricted to $(G \cdot \mu)$ gives the reduced (symplectic) system, since as we have mentioned the symplectic and Poisson reduced structures in this case are compatible. Indeed $(G \cdot \mu)$ is a symplectic manifold with symplectic form (1.17) , where $\xi_{\mathcal{G}^*}(\nu) = \frac{d}{dt} Ad^*_{\exp -t\xi}(\nu)\Big|_{t=0}$, and the Poisson bracket is given by

$$\{f, g\}(\nu) = -\langle \nu, [Df_\nu, Dg_\nu] \rangle.$$

If $H : \mathcal{G}^* \to \mathbf{R}$ is a smooth function the dynamics generated by H is given by

$$\dot{f}(\nu) = \{f, H\}(\nu) = -\langle \nu, [Df_\nu, DH_\nu] \rangle \qquad \text{for all smooth} \quad f : \mathcal{G}^* \to \mathbf{R} \tag{1.19}$$

where the derivatives Df_ν, DH_ν belong to $\mathcal{G}$ as in proposition 1.

Supposing there is an Ad-invariant inner product on $\mathcal{G}$, say $\ll, \gg$, we can then identify $\mathcal{G}^*$ with $\mathcal{G}$ and (1.19) can be written as

$$\ll Df_\nu, \frac{d\nu}{dt} \gg = - \ll \nu, [Df_\nu, DH_\nu] \gg . \tag{1.20}$$

By the Ad-invariance of the inner product we have the following:

Lemma 1

$$\ll \zeta, [\xi, \eta] \gg = \ll \xi, [\eta, \zeta] \gg \qquad \textit{for all} \quad \xi, \eta, \zeta \in \mathcal{G}$$

Proof : The infinitesimal generator for the adjoint action is

$$\frac{d}{dt} Ad_{\exp t\xi}\, \eta \Big|_{t=0} = [\xi, \eta] = ad_\xi \eta,$$

where $ad_\xi : \mathcal{G} \to \mathcal{G}$. By the Ad-invariance of $\ll, \gg$ we have

$$\ll Ad_{\exp t\xi} \zeta, \eta \gg = \ll \zeta, Ad_{\exp -t\xi} \eta \gg .$$

So

$$\ll ad_\xi \zeta, \eta \gg = \ll \zeta, -ad_\xi \eta \gg \Longleftrightarrow \ll [\xi, \zeta], \eta \gg = - \ll \zeta, [\xi, \eta] \gg$$

or

$$\ll \zeta, [\xi, \eta] \gg = \ll \eta, [\zeta, \xi] \gg .$$

Proposition 4 *Let G be a compact Lie group and $H : \mathcal{G}^* \to \mathbf{R}$ a G-invariant function. The Hamiltonian system in $\mathcal{G}^*$, associated to H, is given by the following Euler equation*

$$\dot{\nu} = [\nu, DH_\nu] .$$

Proof : Using last lemma we can write (1.20) as

$$\ll Df_\nu, \dot{\nu} \gg = \ll Df_\nu, [\nu, DH_\nu] \gg$$

for all smooth functions $f : \mathcal{G}^* \to \mathbf{R}$. So the result follows.

□

1.3 On Singular Poisson Reduction

For all reduction results given in the last section it is essential that the reduced space be a manifold. As we mentioned two hypotheses can be assumed in order to ensure this, mamely that S itself is a manifold and that the action of T on S is free and proper. If any of these conditions do not hold then the quotient space S/T may not be a manifold. This means, in the case of symplectic reduction, that the reduced space can fail to be a manifold if the level set of the momentum map is not a manifold. In particular if all $p \in J^{-1}(\mathcal{O}_\mu)$ are regular values of J (usually referred as μ be a regular point) then $J^{-1}(\mathcal{O}_\mu)$ is a manifold and by the properties of the momentum map we have that all group orbits through points of $J^{-1}(\mathcal{O}_\mu)$ have the same dimension as well as all nearby orbits (see Theorem 27.1 of Guillemin and Sternberg [20] and §26, §27).

Singularities in the level set of the momentum map lead to group orbits of smaller dimensions which is equivalent to saying that more symmetries are present, or less symmetry is broken. In fact the symmetry presence in Hamiltonian systems defined on symplectic manifolds is closely related with the singular points of the momentum map as can be seen by following proposition.

Proposition 5 (Singular points of the momentum map) *Let $J : P \to \mathcal{G}^*$ be the momentum map for the action, Φ, of the Lie group G on the symplectic manifold (P, ω) by symplectic maps. Denote by G_{p_0} the isotropy subgroup of p_0, i.e $G_{p_0} = \{g \in G : \Phi(g, p_0) = p_0\}$, and $\mathcal{G}_{p_0}$ its Lie algebra.*

Then p_0 is a singular value of J if and only if $\mathcal{G}_{p_0} \neq \{0\}$. The set of singular points of J is closed and G-invariant.

Proof: p_0 is a singular point of J if and only if the range of DJ_{p_0} is not all $\mathcal{G}^*$. As ω is a nondegenerate two-form the defining expression (1.4) of the momentum map gives that p_0 is a singular point if and only if there is a $\xi \in \mathcal{G}$, with $\xi \neq 0$ such that $\xi_P(p_0) = 0$.

By definition of G_{p_0} we have $\mathcal{G}_{p_0} = \{\xi \in \mathcal{G} : \ \xi_P(p_0) = 0\}$. Thus p_0 is singular if and only if $\mathcal{G}_{p_0} \neq \{0\}$.

In order to prove that the set of singular points of the momentum map is G-invariant note that for $\xi \in \mathcal{G}_{p_0} \neq \{0\}$, $\exp t\xi \in G_{p_0}$ for all $t \in \mathbf{R}$.

Let $s = \Phi(g, p_0)$ and $\xi \in \mathcal{G}_{p_0}$. As

$$\begin{aligned} \Phi(g \ \exp t\xi \, g^{-1}, s) &= \Phi(\exp(tAd_g\xi), s) \\ &= \Phi(g, p_0) = s. \end{aligned}$$

Then we have $\exp(tAd_g\xi) \in G_s$ for all $t \in \mathbf{R}$, that is $Ad_g\xi \in \mathcal{G}_s$.

□

Here we will try to summarize some known reduction results which work with a general constraint set $V \subseteq P$, which is not necessarily either a manifold or a level set of a momentum map. These results can be applied to the particular cases of the singular level sets of momentum maps.

A Poisson reduction procedure should give a reduced space $\hat{V}$ together with a family of "smooth" functions on $\hat{V}$, say $W^\infty(\hat{V})$, which inherits the structure of a Poisson algebra.

The general form for V led to very different approaches to reduction, and in particular to Poisson reduction. As we shall see some of these results are given only in terms of function algebras, which sometimes are not the function algebra of any (reduced) space, while others work in the context of momentum mappings and group actions. All of them can be compared when their domains of applicability overlap. In particular Dirac reduction, reduction by invariants and algebraic reduction are mainly based on function algebras while MMW-reduction and algebraic reduction deal more specifically with group actions and momentum mappings.

Before starting the description of these reductions we would like to point out that in the cases where the reduction is for a set V which is the level set of a momentum map there is no loss of generality in consider only zero level sets since we can extend the results to nonzero level sets. Indeed we can always transform a nonzero level set, $J^{-1}(\mu)$, of a momentum map into the zero level set of a momentum map defined in a larger space by the following method.

Let G act on $P \times \mathcal{O}_\mu$ by its action on P and coadjoint action on $\mathcal{O}_\mu$, $P \times \mathcal{O}_\mu$ is a symplectic manifold where the symplectic form is the sum of the symplectic form on P and the minus coadjoint orbit symplectic form on $\mathcal{O}_\mu$. The momentum map for this action on $P \times \mathcal{O}_\mu$ is just

$$\Phi : P \times \mathcal{O}_\mu \to \mathcal{G}^*$$

$$\langle \Phi(p,\nu), \xi \rangle = \langle J(p), \xi \rangle - \langle \nu, \xi \rangle$$

for all $\xi \in \mathcal{G}$. There is a neighbourhood U of $\mathcal{O}_\mu$ and a diffeomorphism $\beta : P \times U \to P \times U$ with

$$\beta(J^{-1}(\mu) \times U) = \Phi^{-1}(0) \cap (P \times U)$$

such that

$$J(p) = \mu \Longleftrightarrow \Phi(\beta(p,\nu)) = 0 \qquad \text{for all } \nu \in U. \tag{1.21}$$

Among all the reductions Dirac reduction can be considered the most general, in the sense that it does not impose any restrictions on the set V, no momentum map or group action is required for its application, and it can be applied in all the contexts in which the others are , although it does not always produce a reduced Poisson algebra.

The algebraic reduction, proposed by Śniatycki and Weinstein [51], is only defined in the context of group actions with equivariant momentum mappings and always produces a reduced Poisson algebra but not always a reduced space, i.e the reduced Poisson algebra may not be the function algebra of any space.

The reduction by invariants (or universal) reduction, due to Arms, Cushman and Gotay [3], always produces a reduced space and a reduced Poisson algebra of functions on this space.

The MMW-reduction though it does not work for all singular values of the momentum map has as its main aim the construction of a reduced space M_μ while the reduced Poisson algebra arises as the algebra of smooth functions $C^\infty(M_\mu)$ on this space. This reduction is done in the context of symplectic geometry and produces a symplectic reduced space. The reduced Poisson algebra produced by the MMW reduction does not always coincide with the one obtained by the reduction by invariants.

A more detailed treatment of all these reductions, can be found in Wilbour and Arms [6] and Arms, Gotay and Jennings [4].

Consider a variety V as being a topologic space together a set of smooth functions $C^\infty(V) \subseteq C^0(V)$. If $\pi : V \to \check{V}$ is a surjection onto $\check{V}$ we can make $\check{V}$ a quotient variety of V by putting the quotient topology on $\check{V}$ and taking

$$C^\infty(\check{V}) = \left\{ f \in C^0(\check{V}) : \ f \circ \pi = F \text{ for some } F \in C^\infty(V) \right\}.$$

Similarly we can make a subset $\hat{V}$ of a variety V a subvariety by putting the relative topology on $\hat{V}$ and taking $C^\infty(\hat{V}) = W^\infty(\hat{V})$ with

$$W^\infty(\hat{V}) = \left\{ f \in C^0(\hat{V}) : \ f = F\big|_{\hat{V}} \text{ for some } F \in C^\infty(V) \right\},$$

where $W^\infty(\hat{V})$ are the Whitney smooth functions on $\hat{V}$.

MMW-Reduction

The symplectic reduction of last section can be generalized to weakly regular points of the momentum map. We will call this generalization MMW-reduction (Meyer,Marsden and Weinstein) though it is also known as geometric reduction.

Let V be a symplectic manifold with symplectic form ω such that $\operatorname{Ker}\omega$ has constant dimension. This condition allows us to define a distribution on V. Suppose that this distribution is fibrating, i.e there exits a manifold W and a submersion $\pi : V \longrightarrow W$ such that the leaves of the foliation defined by the distribution are the preimages of points in W.

There exists a unique symplectic form $\tilde{\omega}$ on W with $\omega = \pi^*\tilde{\omega}$. The reduced algebra is $C^\infty(W)$, the algebra of smooth functions on the (geometric) reduced space W.

In the case that V is $J^{-1}(\mu)$, the level set of an equivariant momentum map, then $\operatorname{Ker}\omega$ is the tangent space at a point $p \in V$ to the G_μ orbit of p and the reduced space is the one defined in last section $W = V/G_\mu = J^{-1}(\mu)/G_\mu$.

Dirac Reduction

Note that given two functions on V they can be extended arbitrarily off V to functions $f, g \in C^\infty(P)$ but the Poisson bracket $\{f,g\}_P\big|_V$ depends in general on the values of f, g off V.

Let $\mathcal{I}(V)$ be the ideal of functions of $C^\infty(P)$ which vanish on V. The largest class of functions for which the bracket $\{f,g\}_P\big|_V$ does not depend on the values of f, g off V is the normalizer of the ideal $\mathcal{I}(V)$, defined by:

$$N(\mathcal{I}(V)) = \{ f \in C^\infty(P) \ : \ \{f, \mathcal{I}(V)\} \subseteq \mathcal{I}(V) \ \}.$$

Define an observable and an equivalence relation $\sim$ on V respectively by:

- The set of observables $\mathcal{OB}(V)$ is

$$\mathcal{OB}(V) = \{ \ h \in C^\infty(P) \ : \ \{h, N(\mathcal{I}(V))\} \subseteq \mathcal{I}(V) \ \}$$

- Let $p, q \in V$, the equivalence $\sim$ is given by

$$p \sim q \Longleftrightarrow h(p) = h(q) \qquad \text{for all} \qquad h \in \mathcal{OB}(V)$$

The Dirac reduced space is now defined as $\hat{V} = V/\sim$ which is isomorphic to $\mathcal{OB}(V)/\mathcal{I}(V)$. $W^\infty(\hat{V})$ is a Poisson algebra if and only if $\mathcal{I}(V) = N(\mathcal{I}(V))$ and in this case the Poisson bracket is defined by

$$\{\hat{h}, \hat{k}\}_{\hat{V}}(\hat{q}) = \{h, k\}_P(q)$$

for $q \in \pi^{-1}(\hat{q})$ (with $\pi : V \longrightarrow \hat{V}$ the canonical projection) $\hat{k}, \hat{h} \in W^\infty(\hat{V})$, $\hat{q} \in \hat{V}$ and $\{\ ,\ \}_P$ denoting the Poisson bracket on P.

When Dirac's reduction is applied to a V which is the zero level set of an equivariant momentum map for the symplectic action of a Lie group G on (P, ω) we have $\mathcal{I}(V) \subseteq N(\mathcal{I}(V))$. If G is compact then $\hat{J}(\xi) : P \to \mathbf{R}$ is in $N(\mathcal{I}(V))$ and observable functions are G-invariant on V, i.e

$$h \in \mathcal{OB}(V) \iff \{h, \hat{J}(\xi)\}\Big|_V = 0 \;\; \forall \xi \in \mathcal{G}.$$

We also have in this case that $\hat{V}$ can be identified with

$$[W^\infty(V)]^G = \{f \in W^\infty(V) \; : \; f \text{ is } G\text{-invariant on } V\}.$$

Although if $\mathcal{I}(V) \neq N(\mathcal{I}(V))$ then Dirac's reduction fails to produce the symplectic structure. (See Arms, Gotay and Jennings [3] for all these results).

Algebraic reduction

The algebraic reduction only works in the context of group actions with equivariant momentum maps J.

Let $\mathcal{I}(J)$ be the ideal of C^∞ functions generated by the components of an equivariant momentum map J, i.e if $\{\xi_k\}$ is a basis for the Lie algebra $\mathcal{G}$ and $J_k = \langle J, \xi_k \rangle$ then $\mathcal{I}(J) = \{f \in C^\infty : \; f = \sum_k f_k J_k \}$.

The reduced algebra is the subspace of the G-invariant function classes:

$$\mathcal{V} = [C^\infty(P)/\mathcal{I}(J)]^G .$$

It is possible to show that the Poisson algebra structure on $C^\infty(P)$ induces one on $\mathcal{V}$.

In Śniatycki and Weinstein [51] is shown that in the case when G is connected, zero is a regular value of the momentum map J and $V = J^{-1}(0)$ the reduced Poisson algebra may be identified with the Poisson algebra of the reduced space V/G.

Reduction by Invariants (or Universal reduction)

This reduction is due to Arms, Cushman and Gotay [3] and is also known as universal reduction. Let $\mathcal{I}(V)$ be the ideal of smooth functions which vanish on V and $C^\infty(P)^G$ the set of G-invariant functions on P.

The function algebra produced by this reduction is

$$\mathcal{A}_I = \frac{C^\infty(P)^G}{\mathcal{I}(V) \cap C^\infty(P)^G} \tag{1.22}$$

which is a function algebra on a quotient space of V and is also a Poisson algebra with Poisson bracket induced from the one on P/G.

This means that this reduction always produces a reduced space and a reduced Poisson algebra of functions on this space unlike the algebraic reduction.

Note as well that the function algebra produced by this reduction can be smaller than the one obtained by the MMW-reduction although the reduced space given by both reductions is the same. This reduction is also more general than MMW reduction since it is not necessary that V be smooth and symplectic, nor that the rank of the symplectic form on V be constant.

If $V = J^{-1}(\mu)$ then (1.22) can be identified with the algebra of the Witney smooth functions W^{∞} on $M_\mu = J^{-1}(\mu)/G_\mu$:

$$\mathcal{A}_I = W^{\infty}(J^{-1}(\mu)/G_\mu) = C^{\infty}(P)^G/\mathcal{I}(J^{-1}(\mu))^G$$

which is a Poisson algebra where the bracket is given , for $f_\mu, g_\mu \in \mathcal{A}_I$, by

$$\{f_\mu, g_\mu\} = \{f, g\}_{P/G}\Big|_{M_\mu}$$

where f, g are smooth extensions of f_μ, g_μ to P/G. Note that the function algebra produced by MMW reduction is $C^{\infty}(M_\mu)$ which does not always coincide with $W^{\infty}(J^{-1}(\mu)/G_\mu)$ (see theorem 3 of Arms, Cushman and Gotay [3] for conditions under which the algebras do coincide).

All methods of singular reduction presented are rather abstract and not very easy to apply in pratice. In this sense we think it is worth to mention two other (singular) reduction results, due to Arms, Marsden and Moncrief [5] and Sjamaar and Lerman [49], which explore the structure of a (singular) zero level set of a momentum map and treat the singularities under a different point of view.

These works study the reduction of a zero level set of a momentum map by exploiting its symmetries namely by making use of concepts of group theory such as of orbit type and slices.

The main result of Sjamaar and Lerman [49] is that the reduced space $P_0 = J^{-1}(0)/G$ is a disjoint union of symplectic manifolds $(P_0)_{(H)}$, where $(P_0)_{(H)}$ is the stratum of P_0 of orbit type (H), i.e the set of all points of P_0 with isotropy subgroup conjugated to H. Furthermore the Hamiltonian flows of functions in $C^{\infty}(P_0)$, where this set denotes an appropriate Poisson algebra, preserve the symplectic pieces of P_0, that is the restriction of the Hamiltonian flow of a function h in $C^{\infty}(P_0)$ to the stratum $(P_0)_{(H)}$ equals the hamiltonian flow of the function $h|_{(P_0)_{(H)}}$.

It is also proved in this work that each stratum $(P_0)_{(H)}$ can be obtained as a MMW reduced space, say $J'^{-1}(0)/L$ where L is the quotient of the normalizer of H in G, $N_G(H)$, by H, i.e $L = N_G(H)/H$, and J' is the momentum map J restricted to M'_H, where M'_H is the union of the components of the set of points $p \in P$ with isotropy subgroup Σ_p equal to H which intersect $J^{-1}(0)$ non trivially.

As a particular case of Sjamaar and Lerman's results (namely by proof of theorem 2.1) we can get Arms, Marsden and Moncrief [5] main result. The techniques used by these last authors were different techniques of the former authors namely they assumed that P was a Kähler manifold and they used the Liapunov-Schmidt method in order to deal with the structure of $J^{-1}(0)$ near the singular point p_0.

Let us give a short description of this result. Consider $p_0 \in J^{-1}(0)$, V a slice through p_0 such that $T_{p_0}V = (T_{p_0}(G \cdot p_0))^{\perp}$, where $\perp$ denotes the symplectic orthogonal. Let N_{p_0} be a set of nearby points of p_0 such that the isotropy subgroup of all points $p \in N_{p_0}$ is conjugated to the isotropy subgroup, Σ_{p_0}, of p_0.

The result now says that

$$\frac{N_{p_0} \cap J^{-1}(0)}{G} \cong N_{p_0} \cap V \cap J^{-1}(0)$$

where $N_{p_0} \cap V = \{p \in V \ : \ \Sigma_p = \Sigma_{p_0}\}$ and $N_{p_0} \cap V \cap J^{-1}(0)$ is a symplectic space.

We also have that the dynamics on $N_{p_0} \cap J^{-1}(0)$ can be lifted back from the dynamics on $N_{p_0} \cap V \cap J^{-1}(0)$. This is the same of saying that we can reduce to the dynamics on the slice for points of $J^{-1}(0)$ of the same orbit type as Σ_{p_0}.

We will come back to these results in section 1.5 where they follow as particular cases of the main theorem there.

1.4 Local Normal Form for a Momentum Map

The local normal form for a momentum map, due to Marle [30] and Guillemin and Sternberg [20, 19], is a consequence of the isotropic embedding theorem which gives that the local structure of an isotropic embedded manifold is determined up to isomorphism by the so-called symplectic normal bundle. The fact of a group orbit through a point of the zero level set of the momentum map be embedded isotropically is the key ingredient not only for the normal form of the momentum map but as well for the reduction presented in the next section.

Definition 7 *Let $\mathcal{P}$ be a symplectic manifold with symplectic form ω. We say that $i : O \to \mathcal{P}$ is an isotropic embedding when*

$$di_p(T_pO) \subset (di_p(T_pO))^{\perp} \subset T_{i(p)}P,$$

where $\perp$ denotes the orthogonal complement with respect to the symplectic form ω on P , i.e

$$(T_pP)^{\perp} = \{u \in T_pP : \ \omega(u,v) = 0\}.$$

The isotropic embedding theorem states that given an isotropic embedding of a manifold O into a symplectic manifold $\mathcal{P}$ then a small neighbourhood of O is completely determined by the so-called the symplectic normal bundle $\mathcal{Y}$ of O defined as $\mathcal{Y} = \frac{(TO)^{\perp}}{TO}$, where $(TO)^{\perp}$ denotes the symplectic orthogonal of TO. When we are dealing with symplectic group actions on $\mathcal{P}$ the isotropic embedding theorem has a "symmetric" formulation which is equivalent to say that a small neighbourhood of a given isotropic embedding is determined up to a symplectic isomorphism by the characteristic elements in the language of Marle [30]. These are (if the action produces an equivariant momentum map), the momentum map value at p and the representation of the isotropy subgroup G_p of p in the fiber of $\mathcal{Y}$ through p.

Let us state the isotropic embedding theorem as in Guillemin and Sternberg [20], theorem 39.1:

Theorem 1 (Isotropic embedding theorem) *Any isotropic embedding $i : O \to \mathcal{P}$ determines a symplectic normal bundle $\mathcal{Y} \to O$ where $\mathcal{Y} = \frac{(TO)^{\perp}}{TO}$. If i_1 and i_2 are* *two isotropic embeddings of O respectively into $\mathcal{Y}_1$ and $\mathcal{Y}_2$ and if f is a symplectic diffeomorphism of some neighbourhood of $i_1(O)$ into $\mathcal{Y}_2$ such that $i_2 = f \circ i_1$, then f induces a symplectic isomorphism $L_f : \mathcal{Y}_1 \to \mathcal{Y}_2$ of the corresponding normal bundles. Conversely, given any symplectic isomorphism $L : \mathcal{Y}_1 \to \mathcal{Y}_2$, there exists an f with $L = L_f$.*

By the same reference the choice of a connection allows that: given a symplectic normal bundle $\mathcal{Y} \to O$ there is a standard isotropic embedding of O whose symplectic normal bundle is $\mathcal{Y}$, namely the isotropic embedding of O as the zero section in T^*O where T^*O is regarded as a symplectic submanifold of MMW reduced space $T^*\mathcal{P}/Q$ where $Q \cong \mathbf{R}^{2n}$ with $2n$ the dimension of the fiber of $\mathcal{Y}$.

An equivariant version of this theorem can be obtained by applying the equivariant version of Darboux theorem. Let G be a Lie group acting symplectically on $\mathcal{P}$, J an equivariant momentum map for the action of G and O an isotropic embedding. Suppose also that G acts transitively on O and $p \in O$. The representation of G on $\mathcal{Y}$ is completely determined by the representation of G_p on the fiber $Y = \frac{(T_pO)^\perp}{T_pO}$ which by other hand is determined by the action of G_p on $T_p\mathcal{P}$. By the (equivariant) Darboux theorem the symlectic diffeomorphism f can be choosen (locally) G_p equivariant and the isotropic embedding theorem now gives that a G invariant neighbourhood of O can be recover (up to isomorphism) from the action of G_p on the fiber Y.

Let J be an equivariant momentum map for the symplectic action of the Lie group G on $(\mathcal{P}, \omega)$ and $\mathcal{G}_\alpha$ be the Lie algebra of the isotropy subgroup G_α of $\alpha = J(p)$.

Proposition 6 *A group orbit $O = \mathcal{O}_p$ through a point p is an isotropic embedding into $\mathcal{P}$ if and only if $\mathcal{G}_\alpha = \mathcal{G}$.*

Proof: Differentiating with respect to t at $t = 0$ the equivariance condition:

$$J(\Phi_g(p)) = Ad^*_{g^{-1}} J(p)$$

with $g = \exp t\xi$ and $\xi \in \mathcal{G}$ we get

$$\langle dJ\,(\xi_P(p))\,, \eta\rangle = -\langle \xi_{\mathcal{G}^*}(\alpha), \eta\rangle \tag{1.23}$$

where $\xi_{\mathcal{G}^*}(\alpha) = \frac{d}{dt} Ad^*_{\exp -t\eta}(\alpha)\Big|_{t=0}$. The equality (1.23) and the definition condition (1.4) of the momentum map on symplectic spaces give:

$$\langle dJ\,(\xi_P(p))\,, \eta\rangle = \omega\,(\eta_P(p), \xi_P(p)) = \langle \eta_{\mathcal{G}^*}(\alpha), \xi\rangle = -\langle \xi_{\mathcal{G}^*}(\alpha), \eta\rangle. \tag{1.24}$$

As $\mathcal{G}_\alpha = \{\eta \in \mathcal{G} :\ \eta_{\mathcal{G}^*}(\alpha) = 0\}$ then, if $\mathcal{G} = \mathcal{G}_\alpha$ we get from (1.24) that

$$\omega\,(\eta_P(p), \xi_P(p)) = -\langle \xi_{\mathcal{G}^*}(\alpha), \eta\rangle = 0$$

for all ξ and η belonging to $\mathcal{G}$. That is

$$T_p(\mathcal{O}_p) \subset (T_p(\mathcal{O}_p))^\perp.$$

Conversely, if $T_p(\mathcal{O}_p) \subset (T_p(\mathcal{O}_p))^\perp$ then for all $\xi, \eta \in \mathcal{G}$ we have $\omega\,(\eta_P(p), \xi_P(p)) = 0$. This, by (1.24), is equivalent to say that, for all $\xi \in \mathcal{G}$, $\xi_{\mathcal{G}^*}(\alpha) = 0$, that is that $\mathcal{G} = \mathcal{G}_\alpha$. □

Note that although not all the group orbits are isotropic embeddings the procedure of page 25 of reducing the study of a nonzero level set to the zero level set of a momentum map in $\mathcal{P} \times \mathcal{O}_\alpha$ for the given action on $\mathcal{P}$ and the coadjoint action on $\mathcal{O}_\alpha$ allow to reduce the general case, $\mathcal{G} \neq \mathcal{G}_\alpha$, to the case where $\mathcal{G} = \mathcal{G}_\alpha$.

The fiber Y of the symplectic normal bundle can be identified with the space Y of the decomposition of the tangent space to $\mathcal{P}$ at p of Montaldi, Roberts and Stewart [38], which in the case of $\mathcal{G}_\alpha = \mathcal{G}$ reduces:

$$\begin{aligned} W &= Ker\ dJ_p \cap T_p(G \cdot p) \\ Y &= Ker\ dJ_p / W \\ Z &= T_pP / Ker\ dJ_p + T_p(G \cdot p) \end{aligned} \tag{1.25}$$

From the same reference this decomposition has the following properties:

- The symplectic form ω on P induces a G_p-invariant symplectic form on Y, ω_Y, and (Y, ω_Y) is a symplectic vector bundle.
- We can identify the fiber at p of the symplectic normal bundle $\mathcal{Y} = (T\mathcal{O})^\perp / T\mathcal{O}$ with Y.
- Z can be identified with the dual of $T_p\mathcal{O}_p$, so $T_pP \cong Y \oplus (T_p\mathcal{O}_p \oplus (T_p\mathcal{O}_p)^*)$.

The normal form of the momentum map now is a consequence of embedding G/G_p isotropically in an appropriated manifold such that the fiber at p of the symplectic normal bundle for this embedding is the fiber Y of the symplectic normal bundle of the embedding of $\mathcal{O}_p$ into $\mathcal{P}$.

Consider the left identification of T^*G with $G \times \mathcal{G}^*$ (see page 16) and T^*G^- denoting T^*G with the minus canonical symplectic form $(-\omega_0)$. Let (Y, ω_Y) be a symplectic vector space.

The left action of G on G and trivial action on Y lifts to a G action on $G \times \mathcal{G}^* \times Y$, Ψ_L given by

$$\Psi_L(g, (h, \nu, y)) = (gh, \nu, y).$$

The right action of G_p on G and its action on Y, say Φ_Y, lifts to an action on $G \times \mathcal{G}^* \times Y$, Ψ_R, given by

$$\Psi_R(g, (h, \nu, y)) = (hg^{-1}\,,\ Ad_g^*\nu\,,\ \Phi_Y(g, y)) \qquad (h, \nu, y) \in G \times \mathcal{G}^* \times Y.$$

Note that Ψ_L and Ψ_R commute and that from the expressions of the momentum maps for these actions we can conclude that the momentum map for the G action is G_p invariant and that the momentum map for the G_p action is G invariant.

These actions, Ψ_L, Ψ_R are symplectic and there is a well defined symplectic G-action on the twisted bundle $G \times \mathcal{G}^* \times_{G_p} Y$ (see Bredon [9] for definition and results on twisted bundles and associated bundles).

Proposition 7 (Guillemin [20] §41) *Let J_{G_p} be the momentum map for the G_p-action, Ψ_R, on $T^*G^- \times Y$ and J_G be the momentum map for the G action Ψ_L on the same space, where T^*G is identified with $G \times \mathcal{G}^*$ via the left action. Then, for $[c, \beta, y] \in J_{G_p}^{-1}(0)/G_p \subset (G \times \mathcal{G}^*)^- \times_{G_p} Y$, J_G is given by*

$$J_G([c, \beta, y]) = Ad^*_{c^{-1}}\beta.$$

Proof : Let $i^* : \mathcal{G}^* \to \mathcal{G}_p^*$ be the dual of the inclusion $i : \mathcal{G}_p \to \mathcal{G}$. The momentum map for the (right) action of G_p on $T^*G \cong G \times \mathcal{G}^*$ is just $(-i^*)$, as can be easily deduced from example 2, where in equation (1.13) J is $(T_e L_h)^*$. So

$$J_{G_p}^{-1}(0) = \left\{(c, \beta, y) \in (G \times \mathcal{G}^*)^- \times Y : J_Y(y) = -i^*\beta\right\},$$

where J_Y denotes the momentum map for the G_p action on Y.

The action of G_p on $G \times \mathcal{G}^* \times Y$ induces an equivalence relation, $\sim$, given by

$$(c_1, \beta_1, y_1) \sim (c_2, \beta_2, y_2) \iff c_2 = c_1 h^{-1} \quad \beta_2 = Ad^*_h \beta_1 \quad y_2 = \Phi_Y(h, y_1)$$

for some $h \in G_p$. Now as the momentum map for the G action on $T^*G \times Y$ is G_p invariant then there is an action of G on $J_{G_p}^{-1}(0)/G_p$ given by

$$(g', [c, \beta, y]) \mapsto [g'c, \beta, y] \quad \text{for } g' \in G \quad \text{and } [c, \beta, y] \in J_{G_p}^{-1}(0)/G_p.$$

So, for $[c, \beta, y] \in J^{-1}(0)/G_p \subset (G \times \mathcal{G}^*)^- \times_{G_p} Y$, the equivalence class relative to the G_p-action on $J_{G_p}^{-1}(0)$, the momentum map for the G-action, $J_G : (G \times \mathcal{G}^*)^- \times_{G_p} Y \to \mathcal{G}^*$, is

$$J_G([c, \beta, y]) = Ad^*_{c^{-1}}\beta.$$

(See example 2 for the expressions of the momentum map in $G \times \mathcal{G}^*$ coordinates). □

Proposition 8 *Under the same conditions and notations of proposition 7*

1) *$J_{G_p}^{-1}(0)/G_p$ is diffeomorphic to $G \times_{G_p} [(\mathcal{G}/\mathcal{G}_p)^* \times Y]$ (if zero is a regular value of J_{G_p}).*

2) *The embedding of G/G_p (as the zero section) into $G \times_{G_p} [(\mathcal{G}/\mathcal{G}_p)^* \times Y]$ is isotropic, with respect to the induced symplectic form on $G \times_{G_p} [(\mathcal{G}/\mathcal{G}_p)^* \times Y]$, and has $G \times_{G_p} Y$ for its normal bundle.*

Proof: Suppose there is an Ad-invariant inner product on $\mathcal{G}$ (which is always true if G is compact or more generally if G is semisimple). Then $\mathcal{G}$ splits, Ad_{G_p} invariantly, as $\mathcal{G} = (\mathcal{G}/\mathcal{G}_p) \oplus \mathcal{G}_p$ and this splitting induces the following $\mathcal{G}^* = (\mathcal{G}/\mathcal{G}_p)^* \oplus \mathcal{G}_p^*$ where $(\mathcal{G}/\mathcal{G}_p)^*$ can be identify with ker i^*. From the last proposition

$$J_{G_p}(c, \nu, \beta, y) = J_Y(y) + \beta \qquad (c, \nu, \beta, y) \in (G \times (\mathcal{G}/\mathcal{G}_p)^* \times (\mathcal{G}_p)^*) \times Y$$

If zero is a regular value for J_{G_p} then

$$\begin{aligned} \phi : G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y &\rightarrow J_{G_p}^{-1}(0) \subset G \times (\mathcal{G}/\mathcal{G}_p)^* \times \mathcal{G}_p^* \times Y \\ (g, \nu, y) &\mapsto (g, \nu, J_Y(y), y) \end{aligned}$$

gives the desired G_p-equivariant diffeomorphism.

For 2) note that the symplectic form in $T^*G^- \times Y$ is $\omega_Y - \omega_0$ and ω_0 in $G \times \mathcal{G}^*$ (left) coordinates (see Abraham and Marsden [1] pg.315) is just

$$\begin{aligned} \omega_{0(c,\beta)}((TL_cC_1, \mu_1), (TL_cC_2, \mu_2)) &= \langle \mu_2, C_1 \rangle - \langle \mu_1, C_2 \rangle + \langle \beta, [C_1, C_2] \rangle \\ &= \langle \mu_2, C_1 \rangle - \langle \mu_1, C_2 \rangle - \langle ad^*_{C_2}\beta, C_1 \rangle \end{aligned}$$

for $(T_eL_cC_i, \mu_i) \in T_{(c,\beta)}(G \times \mathcal{G}^*) = T_cG \times \mathcal{G}^*$ $(i = 1, 2)$ and $C_i \in \mathcal{G}$. So the induced symplectic form on $G \times \mathcal{G}^* \times Y$ is just $\omega_Y - \omega_0$ which in coordinates terms , with $y_1, y_2 \in T_yY$, is given by

$$\begin{aligned} \omega_{Y_y}(y_1, y_2) - \omega_{0_{(c,\beta)}}((TL_cC_1, \mu_1), (TL_cC_2, \mu_2)) = \\ = \omega_{Y_y}(y_1, y_2) + \langle \mu_1, C_2 \rangle - \langle \mu_2, C_1 \rangle - \langle \beta, [C_1, C_2] \rangle \\ = \omega_{Y_y}(y_1, y_2) + \langle \mu_2 + ad^*_{C_2}\beta, C_1 \rangle - \langle \mu_1, C_2 \rangle \end{aligned}$$

As G/G_p embedes as the zero section of $\frac{G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y}{G_p}$ and G_p only fixes the zero element of the tangent spaces at zero of $(\mathcal{G}/\mathcal{G}_p)^*$ and Y, the symplectic form above on $G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y$ gives the following isomorphisms

$$\begin{aligned} T\left(\frac{G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y}{G_p}\right) &\cong T(G/G_p) \times (\mathcal{G}/\mathcal{G}_p)^* \times Y \\ T(G/G_p) &\cong T(G/G_p) \times \{0\} \times \{0\} \\ T(G/G_p)^\perp &\cong T(G/G_p) \times \{0\} \times Y \end{aligned}$$

that is

$$T(G/G_p) \subset T(G/G_p)^\perp \subset T\left(\frac{G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y}{G_p}\right)$$

which means that G/G_p isotropically embedds as the zero section of $\frac{G \times ((\mathcal{G}/\mathcal{G}_p)^* \times Y)}{G_p}$ and has $G \times_{G_p} Y = \frac{T(G/G_p)^\perp}{T(G/G_p) \cap T(G/G_p)^\perp}$ for its normal bundle.

□

We now have two isotropic embeddings of $\mathcal{O}_p$:

$$i_1 : \mathcal{O}_p \rightarrow G \times_{G_p} ((\mathcal{G}/\mathcal{G}_p)^*) \times Y) \qquad \text{and} \qquad i_2 : \mathcal{O}_p \rightarrow \mathcal{P}$$

where i_2 is just the inclusion. The symplectic diffeomorphism f is given by

$$f : G/G_p \rightarrow \mathcal{O}, \qquad gG_p \mapsto \Phi_g(p) = \Phi(g, p)$$

where Φ denotes the action of G on $\mathcal{P}$.

From the above propositions the isotropic embedding theorem now gives the normal form for the momentum map J for the Φ action of G on $\mathcal{P}$:

Theorem 2 (Normal form of a momentum map) *Let $\mathcal{O}_p = G \cdot p$ be isotropically embedded into P and J an equivariant momentum map for the G action on P. Let G_p be the isotropy subgroup of $p \in P$ and Y the fiber at p of the symplectic normal bundle of $\mathcal{O}_p$ with $J(\mathcal{O}_p) = \alpha$. Then, in an invariant neighbourhood of the group orbit $\mathcal{O}_p$, J is given by*

$$J([c, \nu, y]) = \alpha + Ad^*_{c^{-1}}(\nu + J_Y(y)) \qquad [c, \nu, y] \in G \times_{G_p} (\mathcal{G}/\mathcal{G}_p)^* \times Y).$$

1.5 Hamiltonian Slice Reduction

In this section we establish a local reduction for symmetric Hamiltonian systems in a neighbourhood of a group orbit, which works even for singular points of the momentum map and produces a reduced manifold which is isomorphic to a Poisson variety given by a quotient.

The main feature of the reduction presented here, which we call slice reduction, resides in the possibility of the dynamics study be partially reduced to a system in coordinates of the symplectic slice and the coadjoint group orbit which represents the dynamics as a coupling of "vibrational" motions on the symplectic slice and generalized rigid body motions on the coadjoint group orbit.

Studies of the dynamics in a neighbourhood of a group orbit $\mathcal{O}_p$ for general symmetric dynamical systems can be found in the works of Krupa [23] and Field [15, 13, 14]. Krupa's work gives that the dynamics of an equivariant vector field with respect to the action of a compact Lie group G on $\mathbf{R}^n$, in an invariant neighbourhood of a group orbit $\mathcal{O}_p$, reduces to the dynamics of an G_p equivariant vector field defined in the fiber through p of the normal space of the group orbit $\mathcal{O}_p$. That is if $u(t)$ is a trajectory in an invariant neighbourhood of a group orbit of the G equivariant vector field X then $u(t) = \gamma(t)y(t)$ where $\gamma(t)$ is a smooth curve of elements of G and $y(t)$ is a trajectory of a G_p equivariant vector field defined in the fiber through p of the normal bundle $N(\mathcal{O}_p)$ of the group orbit, where G_p denotes the isotropy subgroup of p (see theorem 2.2 of Krupa [23]).

Krupa and Field's works cited above, reduce, using different techniques, the study of the dynamics in an invariant neighbourhood of a group orbit of a G equivariant vector field to the dynamics of a G_p equivariant vector field in a slice, where $G_p \subset G$. The key assumptions of these works are precisely the ones which gives the existence of a slice S_p through any point of the group orbit $G \cdot p$ under consideration, namely the hypotheses of the manifold in question be a Riemannian manifold and the group G being a compact group. A slice S_p through any point $p \in P$ is a submanifold of P cointaining p such that:

1) S_p is closed in $G(S_p)$.

2) $G_p(S_p) = S_p$.

3) $G(S_p)$ is an open neighbourhood of the group orbit $G \cdot p$.

4) $gS_p \cap S_p \neq \emptyset$ if and only if $g \in G_p$.

where G_p denotes the isotropy subgroup of p (see Guillemin and Sternberg [20]).

Summarizing the main result of the works referred we have:

- For each $y \in G \cdot p$, $z \in S_p$ the trajectory Φ of the G equivariant vector field X in a neighbourhood U of the group orbit $G \cdot p$ through z satisfies:

i) $\Phi_z(t) = \gamma_z(t)h_z(t)$, where $\gamma_z(t) \in C(G_z)$ and for all $z \in S_y$, $h_z(t) \in S_y$. Here $C(G_z)$ denotes the centralizer of the isotropy subgroup G_z of z. (See Field [15]).

Obviously this reduction for general symmetric dynamical systems can also be carried out for Hamiltonian systems but is also natural to take advantages of particularities of this kind of systems namely the symplectic structure and the known integrals (that is of the momentum map). It is also expected that the reduced vector field be Hamiltonian in the symplectic or Poisson sense.

When the symplectic form is given in terms of a Riemannian metric the slice S_p at any point p of the group orbit $\mathcal{O}_p$ can be chosen such that $T_pS_p = [T_p(\mathcal{O}_p)]^{\perp}$ where $\perp$ denotes the orthogonal. In general we work with symplectic orthogonal (i.e orthogonal with respect to the symplectic form) and as we will see we single out the subspace Y, which we called symplectic slice, of the decomposition of Montaldi *et al.* (1.25), which is a symplectic vector space contained in T_pS_p when a Riemannian metric is present.

When we restrict our result to symmetric Hamiltonian vector fields which verify the conditions of applicability of Krupa and Field results we see we get extra information for the reduced system, in particular the reduced system is given in symplectic slice coordinates and rigid body coordinates, i.e coordinates of the dual of the Lie algebra $\mathcal{G}$ (see proposition 9 below).

1.5.1 Reduced space

We give a reduction procedure for the space $P_\mu = J^{-1}(\mu)/G_\mu$, with $\mu \in \mathcal{G}^*$ beloging to a neighbourhood of a given $\alpha \in \mathcal{G}^*$ and J a momentum map for the action of a compact Lie group G on P. Although we will restrict hereafter to the case when $\mathcal{G} = \mathcal{G}_\alpha$, or with no loss of generality when $\alpha = 0$, this is not a very heavy restriction since we can always reduce, by the so-called shifting trick (see section 1.3 page 25), the study of a nonzero level set of a momentum map to the study of a zero level set of another momentum map.

The main theorem of this section, theorem 3, is obtained by combining the normal form theorem with techniques of endowing a subset of a Poisson variety with a Poisson structure. There is no assumption of regularity of the momentum map J at α and so $J^{-1}(\mu)/G_\mu$ can be not a symplectic manifold but as we will see it is isomorphic to a Poisson manifold.

By the normal form theorem for a momentum map we have that the momentum map is obtained, in a G invariant neighbourhood of the group orbit $\mathcal{O}_p$, by considering suitable actions of G and G_p in $(T^*G)^- \times Y$ where Y is the fiber at p of the so-called normal bundle of the group orbit. This space Y is called the symplectic slice since if we consider that the symplectic form at $\mathcal{P}$ is given by a G invariant inner product and $\mathcal{G} = \mathcal{G}_\alpha$ then the orthogonal complement to the tangent space at p to the group orbit

$G \cdot p$ is just $Y \oplus Z$ of the decomposition (1.25), being so Y the symplectic part of this tangent space.

Arms *et al.* [3] give how to make a subset of a Poisson variety into a Poisson subvariety.

A Poisson variety $(P, \{\,,\,\}_P)$ is considered to be a topological space P with a set $C^\infty(P)$ of real-valued functions $f : P \to \mathbf{R}$ and a Poisson structure $\{\,,\,\}_P$ on P.

A quotient space $\bar{P}$ of a Poisson variety P is a Poisson variety where the topology in $\bar{P}$ is the quotient topology and the set of smooth functions $C^\infty(\bar{P})$ is

$$C^\infty(\bar{P}) = \left\{ f : \bar{P} \to \mathbf{R} \mid f \circ \pi = F \quad \text{for some} \quad F \in C^\infty(P) \right\},$$

with $\pi : P \to \bar{P}$ denotes the quotient projection, and the Poisson structure on $\bar{P}$, $\{\,,\,\}_{\bar{P}}$, is induced from the one on P:

$$\{f, g\}_{\bar{P}}(\pi(m)) = \{F, G\}_P(m)$$

where F, G are extensions of f and g respectively to P, i.e $f \circ \pi = F$ and $g \circ \pi = G$.

In the case where $\bar{P} = P/G$ with G a Lie group acting on P then the smooth structure $C^\infty(\bar{P})$ is isomorphic to the set of G invariant smooth functions $C^\infty(P/G) = (C^\infty(P))^G$.

A subset V of a Poisson variety P can be made into a Poisson subvariety of P by:

- Taking the topology on V to be the relative topology.
- Taking for the smooth structure in V the set of Whitney smooth functions in V defined as:

$$W^\infty(V) = \left\{ f : V \to \mathbf{R} \mid \; f = F\big|_V \quad \text{for some} \quad F \in C^\infty(P) \right\}.$$

- Taking for the Poisson structure the one obtained by restriction of the Poisson structure on P, i.e

$$\{f, g\}_V = \{F, G\}_P\big|_V.$$

In particular if we take $V = \bar{V}/G$ as a subset of the Poisson variety P/G then $W^\infty(V)$ is the set of functions f for which there is a function of $C^\infty(P/G)$ such that its restriction to V is f. By definition of $C^\infty(P/G)$ this is equivalent to say that $W^\infty(V)$ is the set of functions $f : \bar{V}/G \to \mathbf{R}$ for which there is a G-invariant F such that the restriction of F to $\bar{V}$ equals $f \circ \pi$, where π is the orbit projection $\pi : \bar{V} \to \bar{V}/G$. By the above reference we have that in this case $W^\infty(V)$ is isomorphic to $(C^\infty(P))^G \,/\, \left(\mathcal{I}(\bar{V})\right)^G$ where $\left(\mathcal{I}(\bar{V})\right)^G$ is the ideal of G invariant functions which vanish on $\bar{V}$.

Following step by step the normal form construction for a momentum map it is obvious which smooth structures and Poisson structures shall be taken.

The normal form has been obtained by considering actions of G and G_p on $M = G \times \mathcal{G}^* \times Y$. If J_{G_p} denotes the momentum map for the action of G_p on M we first reduced at zero and obtained for the reduced space $J_{G_p}^{-1}(0)/G_p = M_0^p$. This reduced space is a Poisson variety with the smooth structure given by the set $W^\infty(M_0^p)$ of functions $f : M_0^p \to \mathbf{R}$ for which there is G_p invariant smooth function F on M such that $f \circ \pi_{G_p} = F|_{J_{G_p}^{-1}(0)}$ where $\pi_{G_p} : J_{G_p}^{-1}(0) \to J_{G_p}^{-1}(0)/G_p$ denote the G_p orbit projection. The Poisson bracket on M_0^p is induced from the one on M given by

$$\{f,g\}_{M_0^p} = \{F,G\}_M|_{J_{G_p}^{-1}(0)}$$

where $F, G \in C^\infty(M))^{G_p}$ are smooth extensions of f, g to M.

Furthermore G acts in this reduced space M_0^p and its action has a momentum map, say $K : M_0^p \to \mathcal{G}^*$. We can reduce even more with respect to the G action on M_0^p and obtain $K^{-1}(\mathcal{O}_\mu)/G = M_\mu$. We can endow this set with a smooth structure given by the set $W^\infty(M_\mu)$ of functions $h : M_\mu \to \mathbf{R}$ for which there is a G invariant smooth function H on M_0^p such that $h \circ \pi = H|_{K^{-1}(\mathcal{O}_\mu)}$, where $\pi : K^{-1}(\mathcal{O}_\mu) \to K^{-1}(\mathcal{O}_\mu)/G$ denotes the G orbit projection . The Poisson bracket is defined from the Poisson bracket on M_0^p as

$$\{F,H\}_{M_0^p}|_{K^{-1}(\mathcal{O}_\mu)} = \{f,h\}_{M_\mu}$$

Let us now enunciate the main result.

Theorem 3 *There exists a G-invariant neighbourhood M of $\mathcal{O}_p$ in P and a G_p-invariant neighbourhood U of 0 in Y such that for all μ sufficiently close to 0 in $\mathcal{G}^*$ the reduced space*

$$M_\mu = \frac{J^{-1}(\mu) \cap M}{G_\mu}$$

is isomorphic as a Poisson variety to

$$\frac{\Psi^{-1}(0) \cap (G \cdot \mu \times U)}{G_p}$$

where $G \cdot \mu = \mathcal{O}_\mu$ is the coadjoint orbit through μ and $\Psi : \mathcal{G}^* \times Y \to \mathcal{G}_p^*$ is a G-equivariant map such that its restriction to $\mathcal{O}_\mu \times Y$ is the momentum map for the action of G_p on $\mathcal{O}_\mu \times Y$.

Proof: By the normal form theorem, in a neighbourhood M of the orbit $\mathcal{O}_p$ we have

$$J^{-1}(\mathcal{O}_\mu) = \{[g,\nu,y] \in \frac{G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y}{G_p} \;:\; Ad^*_{g^{-1}}(\nu + J_Y(y)) = \mathcal{O}_\mu\} \tag{1.26}$$

for μ sufficiently close to zero in $\mathcal{G}^*$.

Let $\phi : (\mathcal{G}/\mathcal{G}_p)^* \times Y \to \mathcal{G}^*$ be a G_p equivariant map defined by $\phi : (\nu, y) \mapsto (\nu + J_Y(y))$. The set (1.26) is then given by

$$J^{-1}(\mathcal{O}_\mu) = \{[g,\nu,y] \in \frac{G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y}{G_p} \;:\; (\nu,y) \in \phi^{-1}(\mathcal{O}_\mu)\}$$

So

$$\frac{J^{-1}(\mathcal{O}_\mu)}{G} = \{[1,\nu,y] \in \frac{G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y}{G_p} \ : \ (\nu,y) \in \phi^{-1}(\mathcal{O}_\mu)\}.$$

That is $\frac{J^{-1}(G\cdot\mu)}{G} \cong \frac{\phi^{-1}(G\cdot\mu)}{G_p}$.

Consider the map

$$\begin{aligned} \Psi : \mathcal{G}^* \times Y &\to \mathcal{G}_p^* \\ (\beta, y) &\mapsto -i^*\beta + J_Y(y) \end{aligned}$$

its restriction to $\mathcal{O}_\mu \times Y$ is just the momentum map for the G_p action on $\mathcal{O}_\mu \times Y$. Then $\Psi^{-1}(0) \cong \phi^{-1}(G\cdot\mu) \subset (\mathcal{G}/\mathcal{G}_p)^* \times Y$ and

$$\frac{\Psi^{-1}(0)}{G_p} \cong \frac{\phi^{-1}(\mathcal{O}_\mu)}{G_p} \cong \frac{J^{-1}(\mathcal{O}_\mu)}{G} \cong \frac{J^{-1}(\mu)}{G_\mu}$$

By taking suitable neighbourhoods we have that M_μ is isomorphic to

$$\frac{\Psi^{-1}(0) \cap (G\cdot\mu \times U)}{G_p}.$$

Let us now define as before the smooth structure in M_μ which makes it isomorphic as a Poisson variety to $\dfrac{\Psi^{-1}(0) \cap (G\cdot\mu \times U)}{G_p}$.

Denote by:

- $W^\infty(J^{-1}(\mathcal{O}_\mu)/G)$ the set of functions $f : J^{-1}(\mathcal{O}_\mu)/G \to \mathbf{R}$ for which there is a G invariant function F of $W^\infty\left(J_{G_p}^{-1}(0)/G_p\right)$ such that $f \circ \pi_G = F\rceil_{J^{-1}(\mathcal{O}_\mu)}$ where $\pi_G : J^{-1}(\mathcal{O}_\mu) \to J^{-1}(\mathcal{O}_\mu)/G$ denotes the orbit projection.
- $W^\infty(\phi^{-1}(\mathcal{O}_\mu)/G_p)$ the set of functions $f : \phi^{-1}(\mathcal{O}_\mu)/G_p \to \mathbf{R}$ for which there is a G_p invariant function F of $C^\infty((\mathcal{G}/\mathcal{G}_p)^* \times Y)$ such that $f \circ \pi'_{G_p} = F\rceil_{\phi^{-1}(\mathcal{O}_\mu)}$ where $\pi'_{G_p} : \phi^{-1}(\mathcal{O}_\mu) \to \phi^{-1}(\mathcal{O}_\mu)/G_p$ denotes the orbit projection.
- $W^\infty\left(J_{G_p}^{-1}(0)/G_p\right)$ the set of functions $f : J_{G_p}^{-1}(0)/G_p \to \mathbf{R}$ for which there is a G_p invariant function F of $C^\infty(G \times \mathcal{G}^* \times Y)$ such that $f \circ \pi_{G_p} = F\rceil_{J_{G_p}^{-1}(0)}$ where $\pi_{G_p} : J_{G_p}^{-1}(0) \to J_{G_p}^{-1}(0)/G_p$ denotes the orbit projection.

Let us prove that $W^\infty\left(\frac{J^{-1}(\mathcal{O}_\mu)}{G}\right)$ and $W^\infty\left(\frac{\phi^{-1}(\mathcal{O}_\mu)}{G_p}\right)$ are isomorphic as Poisson algebras.

Consider the following diagram:

$$\begin{tikzcd} \phi^{-1}(\mathcal{O}_\mu) \arrow[r, "i_e"] \arrow[d, "\pi'_{G_p}"'] & G \times_{G_p} \phi^{-1}(\mathcal{O}_\mu) \arrow[d, "\pi_G"] \\ \frac{\phi^{-1}(\mathcal{O}_\mu)}{G_p} \arrow[r, "h"'] & \frac{G \times_{G_p} \phi^{-1}(\mathcal{O}_\mu)}{G} \end{tikzcd}$$

$$\begin{tikzcd}
\phi^{-1}(\mathcal{O}_\mu) \arrow[r, "i_e"] \arrow[d, "\pi'_{G_p}"'] & G \times_{G_p} \phi^{-1}(\mathcal{O}_\mu) \arrow[d, "\pi_G"] \\
\frac{\phi^{-1}(\mathcal{O}_\mu)}{G_p} \arrow[r, "h"'] & \frac{G \times_{G_p} \phi^{-1}(\mathcal{O}_\mu)}{G}
\end{tikzcd}$$

Where i_e is the G_p equivariant embedding $a \mapsto [e, a]$ and h is the isomorphism given by:

$$h : G_p \cdot a \mapsto G \cdot [e, a] \qquad h^{-1} : G \cdot [g, a] \mapsto G_p \cdot a,$$

and π'_{G_p}, π_G the respective orbit projections.

1. Proof of $W^\infty\left(\frac{J^{-1}(\mathcal{O}_\mu)}{G}\right) \subset W^\infty\left(\frac{\phi-1(\mathcal{O}_\mu)}{G_p}\right)$:

We need to prove that if $f \in W^\infty\left(\frac{J^{-1}(\mathcal{O}_\mu)}{G}\right)$ then $f \circ h \in W^\infty\left(\frac{\phi-1(\mathcal{O}_\mu)}{G_p}\right)$, where $h : \frac{J^{-1}(\mathcal{O}_\mu)}{G} \to \frac{\phi-1(\mathcal{O}_\mu)}{G_p}$ is the isomorphism. By definition, this is :

$$f \circ h \circ \pi'_{G_p} = F\big|_{\phi^{-1}(\mathcal{O}_\mu)} \qquad \text{for some} \qquad F \in C^\infty\left((\mathcal{G}/\mathcal{G}_p)^* \times Y\right)^{G_p}.$$

Let $f \in W^\infty\left(\frac{J^{-1}(\mathcal{O}_\mu)}{G}\right)$. By definition, this means that there is a G invariant function F belonging to $W^\infty(G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y)$ such that $f \circ \pi_G = F\big|_{J^{-1}(\mathcal{O}_\mu)}$ and so a G_p invariant function H of $C^\infty(G \times \mathcal{G}^* \times Y)$ for which $F \circ \pi_{G_p} = H\big|_{J^{-1}(0)}$.

If H is a G_p invariant function defined on $G \times \mathcal{G}^* \times Y$ then there is a G_p invariant function $\bar{H} \in C^\infty(\mathcal{G}^* \times Y)$, given by:

$$\bar{H}(\beta) = H(g;\beta) = H_g(\beta) \qquad \text{with} \quad g \in G \quad \text{and} \quad \beta \in \mathcal{G}^* \times Y.$$

As the restriction of H to $G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y$ equals $F \circ \pi_G$ then the restriction of $\bar{H}$ to $(\mathcal{G}/\mathcal{G}_p)^* \times Y$ is a G_p invariant function $C^\infty((\mathcal{G}/\mathcal{G}_p)^* \times Y)$, and

$$\bar{H}\big|_{(\mathcal{G}/\mathcal{G}_p)^* \times Y} = H\big|_{\phi^{-1}(\mathcal{O}_\mu)}.$$

Furthermore F is a G invariant function such that its restriction to $J^{-1}(\mathcal{O}_\mu)$ equals $f \circ \pi_G$, then there is a G_p invariant function $\bar{H}$ of $C^\infty((\mathcal{G}/\mathcal{G}_p)^* \times Y)$ such that

$$f \circ \pi_G = \bar{H}\big|_{\phi^{-1}(\mathcal{O}_\mu)}.$$

Thus for $a \in \phi^{-1}(\mathcal{O}_\mu)$ we have:

$$\begin{aligned} f \circ h \circ \pi'_{G_p}(a) &= (f \circ h)(G_p \cdot a) = f(G \cdot [e, a)) = f \circ \pi_G([e, a]) \\ &= F([e, a]) = H(e, a) = \bar{H}(a). \end{aligned}$$

with $H \in C^\infty((\mathcal{G}7^* \times Y)^{G_p}$. That is $(f \circ h) \in W^\infty\left(\frac{\phi-1(\mathcal{O}_\mu)}{G_p}\right)$.

2. $W^\infty\left(\frac{\phi-1(\mathcal{O}_\mu)}{G_p}\right) \subset W^\infty\left(\frac{J^{-1}(\mathcal{O}_\mu)}{G}\right)$.

We need to prove that if $f \in W^\infty\left(\frac{\phi-1(\mathcal{O}_\mu)}{G_p}\right)$ then $f \circ h^{-1} \in W^\infty\left(\frac{J^{-1}(\mathcal{O}_\mu)}{G}\right)$. By definition, $f \in W^\infty\left(\frac{\phi-1(\mathcal{O}_\mu)}{G_p}\right)$, is equivalent to say that $f \circ \pi'_{G_p} = H\big|_{\phi^{-1}(\mathcal{O}_\mu)}$ for some G_p invariant function $H \in C^\infty\left((\mathcal{G}/\mathcal{G}_p)^* \times_{G_p} Y\right)$. H induces a G_p invariant function , say $\tilde{F}$ of $C^\infty(G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y)$ given by $\tilde{F} \circ i_e = H$ and such that $\tilde{F} \circ i_e\big|_{\phi^{-1}(\mathcal{O}_\mu)} = H\big|_{\phi^{-1}(\mathcal{O}_\mu)}$. $\tilde{F}$ is not necessarily a G invariant function, however as G is compact we can average it over G and get a G invariant function of $C^\infty(G \times \mathcal{G}/\mathcal{G}_p)^* \times Y)$, say F, such that its restriction to $J_{G_p}^{-1}(0)$ equals the restriction of H to $\phi^{-1}(\mathcal{O}_\mu)$. Putting everything together we have, for $[g\,,\,a] \in G \times_{G_p} (\mathcal{G}/\mathcal{G}_p)^* \times Y)$:

$$\left(f \circ h^{-1} \circ \pi_G\right)([g\,,\,a]) = \left(f \circ h^{-1}\right)(G \cdot [g\,,\,a]) = f\,(G_p \cdot a) = f \circ \pi'_{G_p}(a) = H(a)$$

Using the definitions of H, $\tilde{F}$ and F, and the G invariance of F we have:

$$H(a) = \tilde{F}([e\,,\,a]) = F([e\,,\,a]) = F(g \cdot [e\,,\,a]) = F([g\,,\,a])$$

This equality and the previous one give that $f \circ h^{-1}$ belongs to $W^\infty\left(\frac{J^{-1}(\mathcal{O}_\mu)}{G}\right)$.

The same kind of arguments apply for the Poisson brackets.

□

Remark: The proof that the function algebras $W^\infty(\phi^{-1}(\mathcal{O}_\mu)/G_p)$ and $W^\infty(J^{-1}(\mathcal{O}_\mu)/G)$ are isomorphic could also be obtained from the Banach-Stone theorem (see Simmons [44] which states that the homeomorphism between two compact Hausdorff spaces is equivalent to the isomorphism between their corresponding function algebras. In this case the homeomorphism in question is $\frac{\phi^{-1}(\mathcal{O}_\mu)}{G_p} \to \frac{J^{-1}(\mathcal{O}_\mu)}{G}$ given by $G_p(a) \mapsto G \cdot [e, a]$ (see Bredon [9]).

Some imediate consequences can be obtained from theorem 3 for instance, next corollary gives a local version of what is well known for the cotangent bundle reduction (see section 1.2, page 21), where the reduced spaces are coadjoint bundles over P_0 with fiber the coadjoint orbit $\mathcal{O}_\mu$.

Corollary 1 *If $G_p = \{id\}$ and the action of G on $J^{-1}(0)$ is free then the symplectic manifold P_μ for μ near zero is symplectically isomorphic to a bundle over $P_0 = J^{-1}(0)/G$ with fiber the coadjoint orbit $\mathcal{O}_\mu$.*

Proof

For $G_p = \{id\}$ then $\mathcal{G}_p = \{0\}$ and p is a regular point of the momentum map, that is J is regular along $\mathcal{O}_p$.

A neighbourhood M of $\mathcal{O}_p$ is isomorphic to a neighbourhood of $G \times \{0\}$ in $T^*G \times Y$. For regular points Y can be identify with the tangent space at $\mathcal{O}$ to $P_0 = J^{-1}(0)/G$.

As the action is free then for μ near zero P_μ is a manifold and the isomorphism between M_μ and $(U \times G \cdot \mu)$ is symplectic.

□

In particular, for the cotangent bundle case corollary 1 can be extended to all $\mu \in \mathcal{G}^*$.

Corollary 2 *If G_p is finite then P_μ is isomorphic to the quotient of a coadjoint orbit bundle by a finite group G_p.*

Next corollary follows as consequence of the existence of a slice for the action (coadjoint action) of any compact Lie group, particularly from the construction given by proposition 41.1 of Guillemin and Sternberg [20] of this slice. Let us state the main porperties of this construction.

If G is a compact Lie group then for any $\mu \in \mathcal{G}^*$ the set $k^\sharp$ of the elements of $\mathcal{G}^*$ fixed by the center of the isotropy subgroup G_μ of μ is a slice through μ, that is the group orbit $\mathcal{O}_\mu$ intersects $k^\sharp$ transversally at μ, $\mathcal{G}^* = T_\mu\mathcal{O}_\mu \oplus T_\mu k^\sharp$, and for points $\beta \in k^\sharp$ near μ the isotropy subgroup of β in G is contained in G_μ.

By the same proposition we also have that the canonical projection $\mathcal{G}^* \longrightarrow \mathcal{G}^*_\mu$ maps bijectivelly $k^\sharp$ onto $\mathcal{G}^*_\mu$, that is we can canonically imbedded $\mathcal{G}^*_\mu$ in $\mathcal{G}^*$ with complementary space $(\mathcal{G}/\mathcal{G}_\mu)^*$.

Corollary 3 *Under the same conditions and notations of theorem 3 let $\mu \in \mathcal{G}^*$ be in a neighbourhood of $0 \in \mathcal{G}^*$ be such that $G_\mu = G_p$.*

Then M_μ is symplectically isomorphic to $\frac{J_Y^{-1}(k^\sharp)}{G_p}$, where $k^\sharp$ is the set of points of $\mathcal{G}^*$ fixed by the center $C(G_\mu) = C(G_p)$ of G_μ.

Proof:

By theorem 3 M_μ is isomorphic to $\Psi^{-1}(0)/G_p$ where $\Psi(\beta, y) = -i^*(\beta) + J_Y(y) \in \mathcal{G}^*_p$, with $(\beta, y) \in \mathcal{G}^* \times Y$.

As by hypothesis $G_\mu = G_p$ and $k^\sharp$ is mapped bijectively onto $\mathcal{G}^*_\mu = \mathcal{G}^*_p$ then due to the G_p invariance of $k^\sharp$ we have $\Psi^{-1}(0)/G_p$ isomorphic to $J_Y^{-1}(k^\sharp)/G_p$. This is a symplectic isomorphism since $k^\sharp$ is a symplectic vector manifold (note that it is a fixed point space).

□

We would like to relate the results obtained to the works of Arms, Marsden and Moncrief [5] and Sjamaar and Lerman [49]. The main difference between these works and our result is that they only deal with a single level set of the momentum map whereas our result enables us to consider all the momentum values in a neighbourhood of a given one simultaneously.

The main result of Sjamaar and Lerman [49] is that the reduced sapce $P_0 = J^{-1}(0)/G$ can be decomposed into a disjoint union of symplectic spaces $(P_0)_{(H)}$ where H is a subgroup of G and $(P_0)_{(H)} = \left(P_{(H)} \cap J^{-1}(0)\right)/G$, with $P_{(H)}$ denoting the set of points of P with isotropy subgroup conjugate to H (the stratum of P with orbit type (H)).

Arms, Marsden and Moncrief [5] using Liapunov-Schmidt procedure and the assumption that the symplectic form on P is given by a weak Riemannian metric prove that for points with the same symmetry type as p in a neighbourhood of a singular (or nonsingular) point $p \in P$, say N_p, the dynamics on $N_p \cap J^{-1}(0)$ can be reconstructed from the dynamics on the reduced space $\left(N_p \cap J^{-1}(0)\right)/G$, which is just $Y \cap N_p \cap J^{-1}(0)$.

These results are easily obtained from the proof of theorem 3 for the case of $\mu = 0$. Indeed $J^{-1}(0)$ is there given by:

$$\begin{aligned} J^{-1}(0) &= \{[g,\nu,y] \in \tfrac{G\times(\mathcal{G}/\mathcal{G}_p)^*\times Y}{G_p} \ : \ Ad^*_{g^{-1}}(\nu + J_Y(y)) = 0\} \\ &= \{[g,\nu,y] \in \tfrac{G\times(\mathcal{G}/\mathcal{G}_p)^*\times Y}{G_p} \ : \ (\nu, y) \in \phi^{-1}(0)\} = \{0\} \times J_Y^{-1}(0)\} \end{aligned}$$

That is $\frac{J^{-1}(0)}{G} \cong \frac{G\times\{0\}\times J_Y^{-1}(0)}{G_p}$. If one considers now only those points of $J^{-1}(0)$ with isotropy subgroup conjugate to G_p we have from above that it is equal to $G/G_p \times Fix(G_p, Y)$, where $Fix(G_p,Y)$ is the set of elements of Y fixed by G_p. The fixed point set, $Fix(G_p,Y)$, is a symplectic space since Y is a symplectic vector space (see Guillemin and Sternberg [20] Lemma 27.1) and so $\left(P_{(G_p)} \cap J^{-1}(0)\right)/G$ is just $Fix(G_p, Y)$.

This gives that P_0 decomposes into a union of symplectic spaces $(P_0)_{(G_p)}$ if one consideres G_p varying in the (partially ordered) set of conjugacy classes $[G_p]$ of isotropy subgroups of G. Sjamaar and Lerman [49] also proved that this decomposition satisfies the frontier condition:

- $(P_0)_{(H)} \cap \overline{(P_0)}_{(K)} \neq \oslash \Longleftrightarrow (P_0)_{(H)} \subset \overline{(P_0)}_{(K)} \Longleftrightarrow (H) \leq (K)$.

1.5.2 Reduced dynamics

The proof of theorem 3 and the Poisson structures definitions give that the dynamics in a neighbourhood of $\mathcal{O}_p$ can be reduced to the dynamics of a G_p equivariant vector field on $(\mathcal{G}/\mathcal{G}_p)^* \times Y$ which leaves invariant the sets $\phi^{-1}(G\cdot\mu)$ or equivalently to a G_p-equivariant Hamiltonian vector field on $(G\cdot\mu)\times Y$ which leaves $\Psi^{-1}(0)$ invariant. That is for every G_p-invariant function $\widehat{H} : \mathcal{G}^* \times Y \rightarrow R$ its restriction $\widehat{H}_\mu$ to $(G\cdot\mu)\times Y$ defines a G_p-equivariant flow on $G\cdot\mu\times Y$ and then one on $\frac{\Psi^{-1}(0)}{G_p}$. The statement of the theorem 3 is that the flow of a G-invariant function H on M_μ is mapped to the flow on $\dfrac{\Psi^{-1}(0)\cap(G\cdot\mu\times U)}{G_p}$ generated by $\widehat{H}_\mu$.

This suggests that we can partially reduce the dynamics generated by a G-invariant Hamiltonian function H near $\mathcal{O}_p$ on P to the dynamics generated by $\widehat{H}_\mu$ on $(G\cdot\mu)\times Y$. Obviously this reduction is only partial since the dynamics of $\widehat{H}_\mu$ that is relevant for the original system is the one that lies on $\Psi^{-1}(0)$

We can then formulate the following theorem.

Theorem 4 *For every G-invariant function on P there is a G_p-invariant function $\widehat{H}$ on $\mathcal{G}^* \times Y$ such that the isomorphism of theorem 3 maps the flow on M_μ generated by H to the flow on $\frac{\Psi^{-1}(0)\cap(G\cdot\mu\times U)}{G_p}$ generated by $\widehat{H}_\mu$.*

The reduction to a system on $(G\cdot\mu)\times Y$ can always be carried out and the dynamics on $(G\cdot\mu)\times Y$ is represented, under mild hypothesis, as a coupling of internal vibration motions on Y and rigid body dynamics on $(G\cdot\mu)$. Coordinates can be choosen in $\mathcal{G}^*$ and Y such that the reduced system has a particularly simple form being the generalized rigid body dynamics represented by an Euler equation.

The Poisson structure on $\mathcal{G}^*\times Y$ can be taken as the sum of the Poisson structures on $\mathcal{G}^*$ and Y, that is there are coordinates such that the Poisson structure is the sum of the minus Lie-Poisson structure on $\mathcal{G}^*$ (see proposition 1) and the Poisson structure induced by the symplectic form ω_Y on Y.

For compact Lie groups G there is a Ad-invariant inner product $\ll\, ,\, \gg$ on $\mathcal{G}$ which allows to identify $\mathcal{G}$ with $\mathcal{G}^*$. Suppose as well there is a G_p-invariant inner product $(\, ,\,)_Y$ on Y such that the symplectic form is given in terms of this inner product

$$\omega_Y(v,w)=(v,Sw)_Y \qquad \forall v,w\in Y \tag{1.27}$$

where S is a orthogonal linear operator such that $S^2=-I$.

Proposition 9 *The equations of the motion for the dynamics of a G_p-invariant function $\widehat{H}:\mathcal{G}^*\times Y\to\mathbf{R}$ with $\mathcal{G}^*$ identified with $\mathcal{G}$ are*

$$\dot{\xi}= \left[\xi, D_\xi\widehat{H}(\xi,y)\right] \tag{1.28}$$

$$\dot{y}= SD_y\widehat{H}(\xi,y) \tag{1.29}$$

for $y\in Y$ and $\xi\in\mathcal{G}$.

Proof: Taking coordinates (ξ,y) in $\mathcal{G}^*\times Y\cong\mathcal{G}\times Y$ such that the Poisson bracket $\{\, ,\, \}_{\mathcal{G}^*\times Y}$ is the sum of the Poisson brackets of each factor, that is

$$\{f,g\}_{\mathcal{G}\times Y}=\{f,g\}_{\mathcal{G}}+\{f,g\}_Y$$

where $\{\, ,\, \}_{\mathcal{G}}$ denotes the minus Lie Poisson bracket in $\mathcal{G}$ with the Y variable fixed and $\{\, ,\, \}_Y$ the Poisson bracket on Y where the $\mathcal{G}$ variable is kept fixed.

The equations of the motion are

$$\dot{f}(\xi,y)=\{f,\widehat{H}\}_{\mathcal{G}^*\times Y}(\xi,y)$$

or

$$Df_\xi\cdot\dot{\xi}+Df_y\cdot\dot{y}=\{f,\widehat{H}\}_{\mathcal{G}^*\times Y}(\xi,y)$$

Using the inner products on $\mathcal{G}$ and Y we can write last equation as

$$\ll Df_\xi,\dot{\xi}\gg+(Df_y,\dot{y})_Y=\{f,\widehat{H}\}_{\mathcal{G}}(\xi,y)+\{f,\widehat{H}\}_Y(\xi,y)$$

By the form of the symplectic form, (1.27) , on Y and proposition 4 we get the result. □

Remarks:

a) The $J^{-1}(0)$ dynamics generated by H, near $\mathcal{O}_p$, leaves invariant the set $\{0\}\times Y\subseteq\mathcal{G}^*\times Y$ and the dynamics reduces to the $J_Y=0$ dynamics of $\widehat{H}:\{0\}\times Y\to\mathbf{R}$ near $0\in Y$. That is the dynamics of H on $J^{-1}(0)$ and the dynamics of $\widehat{H}(0,y)$ on $J_Y^{-1}(0)$ induce the same flows on the quotient spaces $\frac{J^{-1}(0)}{G}$ and $\frac{J_Y^{-1}(0)}{G_p}$.

b) If the Lie algebra $\mathcal{G}$ is abelian or if $D_\xi \widehat{H}(\xi, y)$ belongs to the center of $\mathcal{G}$ then the Euler equation reduces to $\dot{\xi} = 0$ and the reduced dynamics is given by a family of systems parametrized by $\xi \in \mathcal{G}$.

c) As in corollary 3 if we have $G_\mu = G_p$ then the dynamics on $J^{-1}(\mathcal{O}_\mu)$ generated by H leaves invariant the set $k^\sharp \times Y$ where $k^\sharp$ is the fixed point set of the center $C(G_\mu)$ of G_μ in $\mathcal{G}^*$ and the dynamics is (fully) reduced to the dynamics on $J_Y^{-1}(k^\sharp)$.

As $k^\sharp = \{\mu \in \mathcal{G}^* : \langle \mu, [\eta, \xi] \rangle = 0 \quad \eta \in \mathcal{L}(C(G_\mu)), \, \xi \in \mathcal{G}\}$, the (reduced) Euler equation is $\dot{\eta} = 0$ with η belonging to the Lie algebra $\mathcal{L}(C(G_\mu))$ of $C(G_\mu)$.

So the dynamics on $J^{-1}(\mathcal{O}_\mu)$ generated by H reduces to the dynamics of a family of systems parametrized by the Lie algebra of $C(G_\mu)$, that is to

$$\dot{y} = S D_y \widehat{H}(\eta, y),$$

with $\eta \in \mathcal{L}(C(G_\mu)) = \mathcal{L}(C(G_p))$.

Chapter 2

The Affine Rigid Body

Introduction

In the first chapter symmetric Hamiltonian systems were treated from a theoretical point of view, using Poisson and symplectic geometry. Here we illustrate the theory by studying a particular problem. We choose the affine rigid body which has a very rich geometric structure, but is far from of being completely understood in the geometric context of the first chapter.

An affine rigid body can be defined as a system of material points allowed to have two kinds of motion, rigid rotations and homogeneous deformations, during which all affine relationships are preserved, i.e straight lines and parallel lines remain straight and parallel, though their lengths can vary.

Although the study of affine rigid bodies can be traced back to Newton's study of the shape of the earth, Dirichlet was the first person to suggest looking at motions of a self gravitating homogeneous fluid mass whose internal motions are linear functions (in some inertial frame) of position and which maintains at all times an ellipsoidal figure which can be variable.

The most remarkable contribution to the study of Dirichlet's problem was by Riemann. He gave for the first time a complete description of the several kinds of ellipsoids of equilibrium, known as Riemann's ellipsoids. A historical summary of this subject and an extensive analytical study of the works on ellipsoidal figures of equilibrium by Maclaurin, Jacobi, Dedekind, Riemann and others, can be found in Chandrasekhar [10] and references therein.

Previous works applying the geometric theory of symmetric Hamiltonian systems to the affine rigid body include D. Lewis [24], where a bifurcation result is applied , D. Lewis and J.C.Simo [26], where the energy momentum method is applied to the study of relative equilibria and their stability, and Slawianowski [50] where a group theoretical approach is explored.

Our aim here is to lay the foundations for a treatment of the symmetries of the affine rigid body using the theory presented in Chapter 1, and to show how some well known classical results are a consequence of the symmetries.

We will start by giving a short physical motivation for several kinds of symmetries present in an affine rigid body by looking at Dirichlet's problem. This will show

how the symmetries are related to the so-called body and space coordinates and how the physical meanings of the conserved quantities (elements of the dual of the Lie algebra of the symmetry group) can be viewed as generalizations of angular velocity and circulation.

In section 2.2 the geometry of the affine rigid body is described. The configuration space is taken to be the linear group $GL^+(3)$ and the phase space its cotangent bundle. There is an action of the group $SO(3) \times SO(3)$ on both configuration and phase space where each $SO(3)$ is related to rotations of the body and internal motions (see below or Chandrasekhar [10]). The momentum map for this action is calculated in both body and space coordinates (i.e coordinates $GL^+(3) \times gl(3)^*$ where $gl(3)^*$ is the dual of the Lie algebra of $GL^+(3)$). This momentum map is also the momentum map for the action of the semidirect product of the cyclic group of order 2, $\mathbf{Z}_2$, with $SO(3) \times SO(3)$, where $\mathbf{Z}_2$ acts by transposition on the configuration space and by interchanging the two copies of $SO(3)$. We show also that the momentum map corresponds to the conservation of the angular momentum and circulation by computing these quantities. The relation between these results and the two integrals of motion established by Chandrasekhar [10] and Rosensteel [41] is also made.

In section 2.3 we show how a result of Dedekind for Dirichlet's problem follows as a consequence of the $\mathbf{Z}_2$ symmetry. We explore also the physical interpretation of this symmetry.

In section 2.4 the isotropy lattices for the semidirect product actions of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on the phase space and momentum space, $so(3)^* \times so(3)^*$ are calculated.

The following section, 2.5, shows how the geometric approach we are following enables us to obtain some of Riemann's results without the knowledge of the Hamiltonian. Riemann's theorem and the defining relations of relative equilibria, which are not of 'S-type', in terms of their semi-axes lengths are proved. The symmetries of several kinds of relative equilibria for Dirichlet's problem are also determined.

Section 2.6 is devoted to finding the slice representation of each isotropy subgroup of the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$-action on the phase space in preparation for applications of the slice reduction presented in section 1.5.

In final section the slice reduction is applied to Dirichlet's problem near spherical equilibria. As a consequence of the slice reduction is also established the bifurcation of relative equilibria from an ellipsoidal equilibrium.

2.1 Dirichlet's Problem

The classical approach to Dirichlet's problem is based on the analytical study of the differential equations governing the problem. Here we will analyse Dirichlet's problem as motivation for our geometrical Hamiltonian approach. This enables us to make physical sense of the geometric quantities and group actions for the affine rigid body. We will follow closely the treatment presented in Chandrasekhar [10], chapter 4.

For Dirichlet's problem it is essential to work with two types of reference frames, one which is fixed in space (inertial frame) and another fixed in the body and moving with it (body frame). As we shall see next, the expression of physical quantities with respect to such frames, and the relationship between body and space coordinates, are the key points to understanding the symmetries of this problem.

Let $X = (X_1, X_2, X_3)$ be the coordinates of a position in an inertial frame and $x = (x_1, x_2, x_3)$ the coordinates of the same in the body frame. Assuming that both frames coincide at time $t = 0$, $X(0) = x(0)$, Dirichlet's condition that the internal motions be a linear function of the position (in some inertial frame) can then be written as

$$X(t) = Q(t)x(0) \tag{2.1}$$

where $Q(t)$ is a linear invertible matrix, i.e $Q(t) \in GL^+(3)$.

Any such a matrix Q admits a bipolar decomposition of the form

$$Q = R^T A S \tag{2.2}$$

where R, S are orthogonal matrices ($RR^T = SS^T = \mathbf{I}$) and A is a diagonal matrix, say $A = \mathrm{diag}(a_1, a_2, a_3)$.

Without any loss of generality suppose that $R(0) = S(0) = \mathbf{I}$ and so rewrite equation (2.1) in the form

$$X(t) = R(t)^T A(t) S(t) A_0^{-1} x(0) \tag{2.3}$$

where A_0^{-1} denotes the value of A^{-1} at zero.

The diagonal matrix A_0 has for its entries the semi-axes lengths of the ellipsoid if we consider that the principal axes of the ellipsoid are aligned with the coordinate axes of the body frame, which we always do.

The physical interpretation of the matrices R^T and S is linked, respectively, to the velocity rate of rotation of the body frame with respect to the inertial frame (i.e to the angular velocity) and to vorticial internal motion of the fluid (i.e the vorticity).

Indeed R relates the X coordinates with x coordinates, i.e

$$X = R^T x \tag{2.4}$$

Differentiating this equation we get

$$\dot{X} = \dot{R}^T x + R^T \dot{x} \tag{2.5}$$

Let $\Omega = \dot{R}R^T$, which is a skew-symmetric matrix (easily seen by differentiation of the condition $RR^T = I$), that is an element of the Lie algebra $so(3)$ of $SO(3)$. So, equation (2.5) is equivalent to

$$\dot{X} = R^T(\dot{x} - \Omega x) \tag{2.6}$$

There is an isomorphism between the Lie algebra $\{\mathbf{R}^3, \times\}$ and $\{so(3), [\,,\,]\}$: l : $\mathbf{R}^3 \ni x \mapsto \hat{x}$ (see chapter 1, example 1.) . So, Ω can be identified with a vector $\omega \in \mathbf{R}^3$ which is called the angular velocity of the body frame with respect to the inertial frame. Equation (2.5) can also be written in terms of this isomorphism by differentiating $\hat{X} = R^T \hat{x} R$, that is

$$\begin{aligned} \dot{\hat{X}} &= \dot{R}^T \hat{x} R + R^T \dot{\hat{x}} R + R^T \hat{x} \dot{R} = R^T \left(\Omega^T \hat{x} + \dot{\hat{x}} + \hat{x}\Omega \right) R \\ &= R^T \left(\dot{\hat{x}} + [\hat{x}, \Omega] \right) R \end{aligned}$$

The meaning of S, or more precisely of $\Lambda = \dot{S}S^T$, can be obtained by the relation between the velocity vectors in body and space coordinates.

Let $u = \dot{x}$ denote the "fluid" velocity in body frame coordinates. The vorticity is $\xi = \mathrm{curl}_x u = \nabla \times u$ (where ∇ denotes the gradient operator).

From equation (2.5) and the definition of Ω we have:

$$u = R\dot{X} - R\dot{R}^T x \Longleftrightarrow u = R\dot{X} + \Omega x \tag{2.7}$$

As $X = R^T A S A_0^{-1} x(0) \Longleftrightarrow A_0^{-1} x(0) = S^T A^{-1} R X = S^T A^{-1} x$ then:

$$\begin{aligned} \dot{X} &= \left(\dot{R}^T A S + R^T \dot{A} S + R^T A \dot{S}\right) A_0^{-1} x(0) \\ &= \left(\dot{R}^T + R^T \dot{A} A^{-1} + R^T A \dot{S} S^T A^{-1}\right) x. \end{aligned}$$

Substituting this in (2.7) and applying the Λ definition we have:

$$\begin{aligned} u &= \left(-\Omega + \dot{A}A^{-1} + A\Lambda A^{-1}\right) x + \Omega x = \\ &= \left(\dot{A}A^{-1} + A\Lambda A^{-1}\right) x. \end{aligned} \tag{2.8}$$

The vorticity vector $\xi = (\xi_1, \xi_2, \xi_3) = \nabla \times u$ is just $\xi = \nabla \times (A\Lambda A^{-1})x$, since $\nabla \times (\dot{A}A^{-1})x = 0$. Easy calculations give

$$\xi_i = -\frac{a_j^2 + a_k^2}{a_j a_k}\lambda_i \qquad \text{for} \quad (i \neq j \neq k)$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is the vector identified with Λ. On the other hand, ξ can be identified with a skew-symmetric matrix Σ which is given in terms of the matrix $V = A\Lambda A^{-1}$, by

$$\Sigma = V^T - V. \tag{2.9}$$

Thus we can say that the motion of an affine rigid body consists of a uniform rotation with angular velocity $-\omega$ together with the internal motion u in the frame in which the orientation of the axes of the ellipsoid remain fixed. This internal motion is a vorticial motion with uniform vorticity ξ superposed on an expansion of the form $\dot{A}A^{-1}\, x$.

The classical study of Dirichlet's problem is carried out by studying the equations of the motion governing the fluid, namely the virial equations presented in Chandrasekhar.

Consider an ideal fluid with density $\rho(x, t)$ and isotropic pressure p and suppose that the only force acting, apart from the pressure, is that derived from its own gravitational field. Then the motion satisfies:

$$\rho \dot{u}_i = -\frac{\partial p}{\partial x_i} + \rho \frac{\partial \mathcal{B}}{\partial x_i} \tag{2.10}$$

where $\dfrac{d}{dt}$ denotes the total time derivative and $\mathcal{B}$ is the gravitational potential, that is $\mathcal{B}(x) = G\int_V \dfrac{\rho(x')}{|x - x'|} dx'$. The virial equations of various orders are obtained from equation (2.10) by multiplying it successively by $1, x_j, x_j x_k$ etc., and integrating over the entire volume instantaneously occupied by the fluid. The virial equation of order two provides all that is required for the study of the existence and stability of the ellipsoidal figures of equilibrium of Dirichlet's problem.

Equation (2.10), in an inertial frame, can be written as

$$\rho\dot{U} - \rho\Omega U = -grad\, p + \rho grad\, \mathcal{B} \tag{2.11}$$

where $U = R\dot{X}$.

Rosensteel [41] shows the equivalence of the virial method and the Hamiltonian formulation of the problem. The equations governing the fluid motion are Hamiltonian with Hamilton's function, H, given by

$$H = K + V - pv$$

where K, V are respectively the kinetic and potential energies, p the pressure and $v = \frac{4}{3}\pi a_1 a_2 a_3$ is the constant volume.

2.2 Geometric Formulation and Symmetries

In this section we present a geometric setting for the study of affine rigid bodies which will enable us to apply the techniques developed in Chapter 1. We compute the angular momentum and circulation for the affine rigid body and relate these results with those of Chandrasekhar [10]. The momentum map is determined in several kind of coordinates and it is proved that the components of the momentum map are the angular momentum and the circulation.

Consider two Riemannian manifolds, $\mathcal{B}$ and $\mathcal{S}$, which represent, respectively, a reference configuration and the space in which the body $\mathcal{B}$ moves.

A configuration of the body is a map $\phi : \mathcal{B} \longrightarrow \mathcal{S}$ that is sufficiently smooth, orientation preserving, and invertible. The set $\phi(\mathcal{B})$ is called the current configuration. The configuration space $\mathcal{C}$ is the set of embeddings $\phi : \mathcal{B} \to \mathcal{S}$, that is $\mathcal{C} = \mathrm{Emb}(\mathcal{B}, \mathcal{S})$.

A motion of the body $\mathcal{B}$ is a curve in $\mathcal{C}$, that is a map $\mathbf{R} \ni t \mapsto \phi_t \in \mathcal{C}$ where $\phi_t(X) = \phi(t, X)$ denotes the configuration at time t. Thus a motion is a time-dependent family of configurations.

The derivative of the configuration of a body is called the deformation gradient. More precisely, for a $\mathbf{C}^1$ configuration $\phi : \mathcal{B} \to \mathcal{S}$ the deformation gradient F of ϕ is the tangent map $F = T\phi$.

For the affine rigid body the reference body $\mathcal{B}$ is a subset of $\mathcal{S} = \mathbf{R}^3$. A configuration ϕ is given by $\phi(X) = QX$ where Q is a 3×3 invertible matrix. The deformation gradient in this case is just $F = Q$. As we assume that ϕ preserves the orientation then det $Q > 0$.

Hence the configuration space $\mathcal{C}$ for the affine rigid body case is $GL^+(3)$, the set of 3×3 matrices with (positive) determinant, and the phase space is taken to be the cotangent bundle of the configuration space, $T^*\mathcal{C} = \mathcal{P}$.

Using the bipolar decomposition $Q = R^T AS$ of elements of $GL^+(3)$, the embedding $\phi(X) = QX$ of the reference body X into space can be interpreted as follows. The current configuration is obtained by first rotating the reference body using S, then mapping the body into space with the diagonal matrix A and finally rotating the body in the space by R^T.

Recall that the configuration and phase spaces for the rigid body are, respectively, $\mathcal{C} = SO(3)$, $\mathcal{P} = T^*SO(3)$, that is they are subsets of the phase and configuration spaces for the affine rigid body. In this sense we can say that the rigid body is a particular case of the affine rigid body.

2.2.1 Conserved quantities

Chandrasekhar [10] and Rosensteel [41] show that there are two integrals of the motion for the affine rigid body which correspond to the conservation of the angular momentum and circulation. Here we compute these quantities in the geometric context we are following, and relate our results to the ones of the authors referred to above.

If $X \in \mathcal{B}$ is the position vector of an infinitesimal volume element of $\mathcal{B}$ then the position of that element of volume at time t is $Q(t)X$ and its velocity $\dot{Q}(t)X$. Hence the total angular momentum of the body is:

$$\mathcal{A} = \int_{\mathcal{B}} (QX \times \dot{Q}X)\, \rho(X)\, dX \tag{2.12}$$

where $\rho(X)$ is the density function of $\mathcal{B}$.

Let γ be a closed curve in $\mathcal{B}$. Then $Q\gamma$ is a closed curve in the current configuration which "moves with the fluid". Let $d\mathbf{r}$ be the vector directed along the tangent to γ at every point, with magnitude equal to the element of arc length ds along γ, and $d\mathbf{r}'$ the one for $Q\gamma$. That is if τ is the unit tangent to γ we have $d\mathbf{r} = \tau\, ds$.

As $d\mathbf{r}' = Qd\mathbf{r}$ the circulation of the velocity field $\dot{Q}X$ around $Q\gamma$ is given by:

$$\mathcal{C}_\gamma = \oint_{Q\gamma} \dot{Q}X \cdot d\mathbf{r}' = \oint_{\gamma} \dot{Q}X \cdot Qd\mathbf{r}, \tag{2.13}$$

where the dot indicates the usual inner product of vectors.

Angular momentum

Let $Q \in GL^+(3)$ be $Q = R^T AS$ where R, S are orthogonal matrices and A is diagonal. Define:

$$\Omega = \dot{R}^T R \qquad \Lambda = \dot{S}^T S. \tag{2.14}$$

Both Ω, Λ are $so(3)$ elements, i.e skew symmetric matrices, and can be identified with vectors of $\mathbf{R}^3$ through the isomorphism (1.9) of page 15 (Example 1.). Denote by $\hat{v}$ (or $(v)\hat{}$) the matrix identification of the vector v and by $\hat{\omega} = \Omega$, $\hat{\lambda} = \Lambda$.

Lemma 2 *a)* $\hat{v}\,w = v \times w$ *for all* $v, w \in \mathbf{R}^3$.

b) For any matrix $L \in GL^+(3)$: $\widehat{Lv} = (\det L)\, L^{-T}\hat{v}\, L^{-1}$ *where* $L^{-T} = (L^{-1})^T$.

c) If Σ *is a symmetric matrix then:*

$$\Sigma\,\hat{v} + \hat{v}\,\Sigma = (tr(\Sigma)\mathbf{I} - \Sigma)\hat{}\;v.$$

Proof: The proof of a) and c) is just a calculation, almost imediate if one uses a symbolic programing language. For b), note that for any orthogonal matrix R we have $\widehat{Rv} = R\hat{v}R^T$ and then taking the bipolar decomposition $L = R^T AS$ we get:

$$\widehat{Lv} = (R^T ASv)\hat{} = R^T (ASv)\hat{}\, R. \tag{2.15}$$

If A is a diagonal matrix a short calculation yields $\widehat{Av} = \det(A) A^{-1}\hat{v}A^{-1}$. Applying this to (2.15) we have:

$$\widehat{Lv} = \det(A) R^T A^{-1} (Sv)\hat{}\, A^{-1} R = \det(A) R^T A^{-1} S\hat{v}S^T A^{-1} R = \det(L) L^{-T}\hat{v}L^{-1}.$$

Note that we have used the fact that $\det(A) = \det(L)$.

□

Define the moment of inertia tensors II_0, II (about the origin) of the reference body $\mathcal{B}$ and of the current configuration respectively by:

$$II_0 = -\int_{\mathcal{B}} \hat{X}\hat{X}\,\rho(X)\,dX$$

$$II = -\int_{\mathcal{B}} (\widehat{QX})(\widehat{QX})\,\rho(X)\,dX$$

The components, I_{ij}, of II_0 are:

$$I_{ij} = \begin{cases} \int_{\mathcal{B}} (x_m^2 + x_n^2)\,\rho(X)\,dX & \text{for} \quad i = j \quad \text{and} \quad m \neq n \neq i \\ -\int_{\mathcal{B}} (x_i x_j)\,\rho(X)\,dX & \text{for} \quad i \neq j \end{cases} \tag{2.16}$$

where ρ is the density function. If $\mathcal{B}$ is spherically symmetric then $\rho(X) = \rho(r)$ where $r = \|X\|$ and if the body is homogeneous then the density is independent of X.

For spherically symmetric homogeneous bodies we have $I_{11} = I_{22} = I_{33} = \mu = \frac{2}{5}MR^2$, where R is the radius of the body and M its mass, and $I_{ij} = 0$ for $i \neq j$.

□

Lemma 3 *If* $\mathcal{B}$ *is spherically symmetric then:*

1. $II_0 = \mu\mathbf{I}$.
2. $II = \frac{\mu}{2}(\det(Q))^2 Q^{-T}\left(tr(Q^{-1}Q^{-T})\mathbf{I} - Q^{-1}Q^{-T}\right) Q^{-1}$.

where **I** *denotes the identity matrix and 'tr' the trace.*

Proof:

1. is imediate from the expressions of the I_{ij}.

Let us prove 2. Applying lemma 2 (b) we have :

$$\begin{aligned} II &= -\int_{\mathcal{B}} (\widehat{QX})\,(\widehat{QX})\,\rho(X)\,dX \\ &= -(\det\, Q)^2\, Q^{-T} \left(\int_{\mathcal{B}} \rho(X)\hat{X} Q^{-1} Q^{-T} \hat{X}\, dX \right) Q^{-1} \end{aligned} \tag{2.17}$$

The integral in brackets is of the form $\int_{\mathcal{B}} \rho(X)\hat{X}\, C\, \hat{X}\, dX$, where C is a 3×3 matrix $C = [C_{ij}]$. Calculating this integral we obtain:

$$\begin{aligned} \int_{\mathcal{B}} \rho(X)\,\hat{X}\,C\,\hat{X}\,dX &= -\frac{\mu}{2} \begin{bmatrix} -(C_{22}+C_{33}) & C_{21} & C_{31} \\ C_{12} & -(C_{11}+C_{33}) & C_{32} \\ C_{13} & C_{23} & -(C_{11}+C_{22}) \end{bmatrix} \\ &= \frac{\mu}{2}\left((\mathrm{tr}\, C)\,\mathbf{I} - C^T \right). \end{aligned}$$

Applying this result to (2.17), with $C = Q^{-1}\,Q^{-T}$, we have:

$$II = \frac{\mu}{2}(\det\, Q)^2\, Q^{-T} \left((\mathrm{tr}\,(Q^{-1}\,Q^{-T}))\,\mathbf{I} - (Q^{-1}\,Q^{-T}) \right) Q^{-1}.$$

□

Remark: The moment of inertia tensors for an ellipsoidal reference body and a spherically symmetric one can be easily related. Indeed points on a ellipsoid E of semi-axes lengths a, b, c and points of a sphere of radius R are related by $Y = \frac{1}{R} DX$ where D is a diagonal matrix with entries a, b, c and X, Y belong respectively to the sphere and to the ellipsoid.

Let $II_S = -\int_S \rho(X)\,\hat{X}\,\hat{X}\,dX$ be the moment of inertia tensor of a sphere S of radius R and $II_E = -\int_S \rho(Y)\hat{Y}\,\hat{Y}\,dY$. If J is the Jacobian of the this transformation, the change of coordinates rule for integrals give:

$$\begin{aligned} II_E &= -\int_E \rho(Y)\hat{Y}\,\hat{Y}\,dY = -\frac{1}{R^2}\int_S \rho(X)\,(\widehat{DX}\,\widehat{DX})\,|J|\,dX \\ &= -\frac{1}{R^2}\,|\det\, D\,| \int_S \rho(X)\,(\widehat{DX}\,\widehat{DX})\,dX \end{aligned}$$

Applying part (b) of lemma (2) we get:

$$II_E = \frac{1}{R^2}(\det\, D)^3\, D^{-1}\left[(\mathrm{tr}\, D^{-1})^2\,\mathbf{I} - (D^{-1})^2 \right]\, II_S\, D^{-1}.$$

That is

$$II_E = \frac{\mu}{R^2}(\det\, D)^3 \left[(\mathrm{tr}\, D^{-1})^2 (D^{-1})^2 - (D^{-1})^4 \right].$$

Proposition 10 *If $\mathcal{B}$ is spherically symmetric then the total angular momentum of the current configuration Q has the following expressions:*

1. $\mathcal{A} = II\,\omega - (\det Q)\,Q^{-T}\,II_0\,\lambda.$
2. $\mathcal{A} = \frac{\mu}{2}(\det Q)\,Q^{-T}\left[(\det Q)\left(tr(Q^{-1}Q^{-T})\mathbf{I} - Q^{-1}Q^{-T}\right)Q^{-1}\omega - 2\lambda\right].$
3. $\hat{\mathcal{A}} = \frac{\mu}{2}\left\{\Omega\,(QQ^T) + (QQ^T)\,\Omega - 2Q\,\Lambda\,Q^{-T}\right\}.$

Proof: Let $Q = R^T AS$. Thus

$$\begin{aligned} \dot{Q} &= \dot{R}^T A S + R^T \dot{A} S + R^T A \dot{S} \\ &= \Omega Q - Q\Lambda + R^T \dot{A} S \end{aligned} \tag{2.18}$$

Hence, the expression (2.12) for the angular momentum:

$$\begin{aligned} \mathcal{A} &= \int_{\mathcal{B}} (QX \times \dot{Q}X)\,\rho(X)\,dX = \int_{\mathcal{B}} (QX \times \Omega QX)\,\rho(X)\,dX - \\ &- \int_{\mathcal{B}} (QX \times Q\Lambda X)\,\rho(X)\,dX + \int_{\mathcal{B}} (QX \times R^T\dot{A}SX)\,\rho(X)\,dX \end{aligned}$$

Let us treat each of these integrals separately:
a) As

$$\begin{aligned} QX \times \Omega QX &= -(QX \times (QX \times \omega)) = -\left(QX \times (\widehat{QX}\,\omega)\right) \\ &= -\widehat{QX}\,\widehat{QX}\,\omega \end{aligned}$$

where all the above equalities follow from lemma 2 (a) and $\hat{\omega} = \Omega$, so

$$\int_{\mathcal{B}} (QX \times \Omega QX)\,\rho(X)\,dX = -\left[\int_{\mathcal{B}} \widehat{QX}\,\widehat{QX}\,\rho(X)\,dX\right]\omega = II\,\omega.$$

b) Letting $Y = SX$,

$$\begin{aligned} QX \times Q\Lambda X &= R^T\left(AY \times AS\dot{S}^T Y\right) = R^T\left(AY \times AS\Lambda S^T Y\right) \\ &= (\det A)\,R^T\left(A^{-1}\hat{Y}\,S\Lambda S^T Y\right) = (\det A)\,R^T A^{-1}\left(\hat{Y} \times S\Lambda S^T Y\right) \\ &= -(\det A)\,R^T A^{-1}\left(\hat{Y} \times Y \times S\lambda\right) \\ &= -(\det A)\,R^T A^{-1} S\left(\hat{X} \times X \times \lambda\right) \\ &= -(\det Q)\,Q^{-T}\,\hat{X}\,\hat{X}\,\lambda \end{aligned}$$

Then

$$
\begin{aligned}
-\int_{\mathcal{B}} (Q\,X \times Q\,\Lambda\,X)\,\rho(X)\,dX &= (\det Q)\,Q^{-T}\left(\int_{\mathcal{B}} \hat{X}\,\hat{X}\,\rho(X)\,dX\right)\lambda \\
&= -(\det Q)\,Q^{-T}\,II_0\,\lambda
\end{aligned}
$$

c)

$$
\begin{aligned}
(Q\,X) \times (R^T \dot{A}\,S)\,X &= R^T\left[(A\,S\,X) \times (\dot{A}\,S\,X)\right] = (\det A)\,R^T A^{-1}\,\widehat{SX}\,A^{-1}\,\dot{A}\,S\,X \\
&= (\det A)\,R^T A^{-1}\,S\,\hat{X}\,S^T A^{-1}\,\dot{A}\,S\,X \\
&= (\det A)\,(R^T A^{-1} S)\left[X \times S^T A^{-1}\,\dot{A}\,S\,X\right].
\end{aligned}
$$

Note that $K = S^T A^{-1}\,\dot{A}\,S$ is a symmetric matrix and that for $K = [k_{ij}]$ symmetric a short calculation gives:

$$
(K\,X) \times X = \left(S_1(X) + k_{23}\,(z^2 - y^2),\ S_2(X) + k_{13}\,(x^2 - y^2),\ S_3(X) + k_{12}\,(y^2 - z^2)\right),
$$

where $X = (x, y, z)$ and the S_i's are linear combinations of the inertia moments I_{ij} with $i \neq j$, defined in (2.16). So integrating over $\mathcal{B}$ we have :

$$
\int_{\mathcal{B}} (Q\,X \times R^T \dot{A}\,S\,X)\,\rho(X)\,dX \equiv 0
$$

Finally by a), b), c) we get:

$$
\mathcal{A} = \int_{\mathcal{B}} (QX) \times (\dot{Q}\,X)\,\rho(X)\,dX = II\,\omega - (\det Q)\,Q^{-T}\,II_0\,\lambda.
$$

Also by lemma 3 we have:

$$
\mathcal{A} = \frac{\mu}{2}(\det Q)\,Q^{-T}\left[(\det Q)\left(\mathrm{tr}(Q^{-1}Q^{-T})\,\mathbf{I} - Q^{-1}Q^{-T}\right)Q^{-1}\omega - 2\,\lambda\right]
$$

Taking the matrix form for last expression:

$$
\begin{aligned}
\hat{\mathcal{A}} &= \frac{\mu}{2}(\det Q)\left\{Q\left[(Q^{-1}Q^{-T})(\widehat{Q^{-1}\omega}) + (\widehat{Q^{-1}\omega})(Q^{-1}Q^{-T})\right]Q^T - \frac{2}{\det Q}\,Q\Lambda\,Q^T\right\} \\
&= \frac{\mu}{2}(\det Q)\left\{\frac{1}{\det Q}\left[Q^{-T}Q^T\,\hat{\omega}\,Q\,Q^T + Q\,Q^T\,\hat{\omega}\,Q\,Q^{-1}\right] - \frac{2}{\det Q}\,Q\Lambda\,Q^T\right\} \\
&= \frac{\mu}{2}\left\{\hat{\omega}\,QQ^T + QQ^T\,\hat{\omega} - 2Q\,\hat{\lambda}\,Q^{-T}\right\} = \frac{\mu}{2}\left\{\Omega\,QQ^T + QQ^T\,\Omega - 2Q\,\Lambda\,Q^{-T}\right\},
\end{aligned}
$$

where the first equality follows from lemma 2 b) and c) and the second from b) of the same lemma.

□

Remarks:

i) For a rigid motion we have $A(t)$ and $S(t)$ constant and so $\Lambda = 0$. Hence $\mathcal{A} = II\,\omega$. In this case:

$$II = -\int_{\mathcal{B}} \widehat{R^T X}\, \widehat{R^T X}\, dX = -R^T \int_{\mathcal{B}} \hat{X}\,\hat{X}\, dX\, R = R^T\, II_0\, R$$

So $R\,\mathcal{A} = II_0\, R\omega$, which is the usual relation between the angular momentum in body coordinates and the angular velocity in body coordinates (see Example 1.).

ii) The expressions computed for the angular momentum can be related to those obtained by Chandrasekhar [10] page 74 and Rosensteel [41] §3. First observe that these authors consider for $\Omega = \dot{R}\,R^T$ and $\Lambda = \dot{S}\,S^T$ while we have considered $\Omega = \dot{R}^T R$, $\Lambda = \dot{S}^T S$. Also Chandrasekhar [10] takes $\hat{\omega} = -\Omega$ and we took $\hat{\omega} = \Omega$ (the same for $\hat{\lambda}$ and Λ). Furthermore both authors compute $R\,\mathcal{A}$ which is the angular momentum resolved instantaneously along the principal axes of the deformed body.

Using the expression (3) of proposition 10, and lemma 2 b) we have:

$$\begin{aligned}\widehat{R\mathcal{A}} &= R\,\hat{\mathcal{A}}\,R^T = \frac{\mu}{2}\left\{A^2(R\,\Omega\,R^T) + (R\,\Omega\,R^T)\,A^2 - 2\,A(S\,\Lambda\,S^T)\,A\right\} \\ &= -\frac{\mu}{2}\left\{A^2\,\bar{\Omega} + \bar{\Omega}\,A^2 - 2\,A\,\bar{\Lambda}\,A\right\}\end{aligned}$$

where

$$\bar{\Omega} = -R\,\Omega\,R^T = -R\,\dot{R}^T = \dot{R}\,R^T \quad \bar{\Lambda} = -S\,\Lambda\,S^T = -S\,\dot{S}^T = \dot{S}\,S^T.$$

Note that both Chandrasekhar and Rosensteel define their Ω and Λ to be the same as $\bar{\Omega}$ and $\bar{\Lambda}$ above. Let

$$\begin{array}{lll}\bar{\omega} = (\bar{\omega}_1\,,\,\bar{\omega}_2\,,\,\bar{\omega}_3) & \text{be such that} & \hat{\bar{\omega}} = -\bar{\Omega} \\ \bar{\lambda} = (\bar{\lambda}_1\,,\,\bar{\lambda}_2\,,\,\bar{\lambda}_3) & \text{be such that} & \hat{\bar{\lambda}} = -\bar{\Lambda}\end{array}$$

and $A = \mathrm{diag}\,(a_1\,,\,a_2\,,\,a_3)$.

Then a short calculation gives:

$$R\,\mathcal{A} = \frac{\mu}{2}\left[(a_2^2 + a_3^2)\bar{w}_1 - 2a_2\,a_3\,\bar{\lambda}_1\,,\,(a_1^2 + a_3^2)\bar{w}_2 - 2a_1\,a_3\,\bar{\lambda}_2\,,\,(a_2^2 + a_1^2)\bar{w}_3 - 2a_2\,a_2\,\bar{\lambda}_3\right]$$

which is the expression given in Chandrasekhar [10] page 74.

Circulation

Let γ be a closed curve in the reference body. The circulation of $\dot{Q}\,X$ around $Q\,\gamma$, that is the circulation of the velocity field around the "current" curve, is given by expression (2.13). This expression can be written as:

$$C_\gamma = \int_\gamma \dot{Q}X \cdot Q\,d\mathbf{r} = \int_\gamma Q^T\,\dot{Q}X \cdot d\mathbf{r}.$$

This expression follows from the inner product expression: $u \cdot v = \frac{1}{2}\mathrm{tr}\,(\hat{u}^T \hat{v})$. Indeed, applying lemma 2 and noting that $d\mathbf{r} = \tau\, ds$ we get:

$$\begin{aligned}(\dot{Q}\,X) \cdot (Q\tau)\, ds &= \frac{1}{2}\mathrm{tr}\,((\widehat{\dot{Q}\,X})^T (\widehat{Q\,\tau}))\, ds = \frac{\det Q}{2}\,\mathrm{tr}\,(Q^{-1}(\widehat{\dot{Q}\,X})^T Q^{-T}\,\hat{\tau})\, ds \\ &= Q^T \dot{Q}\, X \ \cdot \tau\, ds.\end{aligned}$$

Here γ is considered to be a closed contour which moves with the fluid , i.e which always consists of the same fluid particles. If γ can be regarded as the boundary of a (oriented) surface S, every point of which lies within the fluid, then we can apply Stokes theorem and relate the circulation of the vector field, say v, with a surface integral, i.e

$$C_\gamma = \oint_\gamma v \cdot d\mathbf{r} = \int\!\!\int_S \mathrm{curl}\, v \cdot n\, dS = \int\!\!\int_S \omega \cdot n\, dS$$

where n denotes the unit normal to S consistent with the orientation of γ, and ω is the vorticity vector. That is we can relate the circulation with the vortex flux through S.

Lemma 4 *Suppose γ is the boundary of a surface $\mathcal{S}$ in $\mathcal{B}$. Then for any matrix L:*

$$\oint_\gamma L\,X \cdot d\mathbf{r} = \oint_\gamma L^A\,X \cdot d\mathbf{r}$$

where $L^A = \frac{1}{2}(L - L^T)$.

Proof: Any matrix L can be written in the form $L = L^S + L^A$ where L^S is the symmetric part. Then

$$\oint_\gamma L\,X \cdot d\mathbf{r} = \oint_\gamma L^S\,X \cdot d\mathbf{r} + \oint_\gamma L^A\,X \cdot d\mathbf{r}.$$

Applying Stokes theorem, we get

$$\oint_\gamma L^S\,X \cdot d\mathbf{r} = \int\!\!\int_S \mathrm{curl}\,(L^S\,X) \cdot n\, dS = \int\!\!\int_S (\nabla \times (L^S\,X)) \cdot n\, dS$$

where ∇ denotes the gradient operator. A short calculation gives that $\nabla \times (L^S\,X) \equiv 0$. hence the result.

□

It follows that $C_\gamma = \oint_\gamma (Q^T\dot{Q})^A \cdot d\mathbf{r} = \frac{1}{2}\oint_\gamma (Q^T\dot{Q} - \dot{Q}^T Q) \cdot d\mathbf{r}$. Let c be the vector such that $\hat{c} = (Q^T\dot{Q})^A$. Then

$$C_\gamma = \oint_\gamma \hat{c}\,X \cdot d\mathbf{r} = \oint_\gamma (c \times X) \cdot d\mathbf{r} = c \cdot \oint_\gamma X \times d\mathbf{r}$$

That is $C_\gamma = c \cdot n_\gamma$ where $n_\gamma = \oint_\gamma X \times d\mathbf{r}$. Note that the vector n_γ depends only on γ and is independent of the motion. Furthermore if we apply Stokes theorem to $C_\gamma = \oint_\gamma (c \times X) \cdot d\mathbf{r}$ we get $C_\gamma = \int\!\!\int_S \mathrm{curl}\,(c \times X) \cdot n\, dS$ and a short calculation gives $\mathrm{curl}\,(c \times X) = 2c$. The vector $2c$ is the Kelvin's circulation vector.

Proposition 11 *The Kelvin's circulation vector is:*

1. $\hat{c} = (Q^T \dot{Q} - \dot{Q}^T Q) = 2Q^T \Omega Q - \left[Q^T Q \Lambda + \Lambda Q^T Q\right]$

2. $c = 2(\det Q) Q^{-1} \omega - \left[\mathrm{tr}(Q^T Q)\mathbf{I} - Q^T Q\right] \lambda$

Proof: Taking, as before, $Q = R^T A S$ from (2.18) we have

$$Q^T \dot{Q} = Q^T \Omega Q - Q^T Q \Lambda + S^T A \dot{A} S$$

and

$$(Q^T \dot{Q} - \dot{Q}^T Q) = 2Q^T \Omega Q - \left[Q^T Q \Lambda + \Lambda Q^T Q\right].$$

Applying lemma 2 b) we get the vector expression of $\hat{c}$.

□

Remark: Chandrasekhar [10], page 74, gives the Kelvin's circulation vector on body reference frame coordinates, that is $\widehat{S c} = S \hat{c} S^T$. Using the same notation as in the last remark we have:

$$\begin{aligned}
\widehat{S c} = S \hat{c} S^T &= (2 A R \Omega R^T A) - (A^2 S \Lambda S^T + S \Lambda S^T A^2) \\
&= -(2 A \bar{\Omega} A) + (A^2 \bar{\Lambda} + \bar{\Lambda} A^2) \\
&= -A \left[2 \bar{\Omega} + A \bar{\Lambda} A^{-1} + A^{-1} \bar{\Lambda} A\right] A \\
&= -2 A \left[\bar{\Omega} - \left((A \bar{\Lambda} A^{-1})^T - A \bar{\Lambda} A^{-1}\right)\right] A.
\end{aligned}$$

The expression $\Sigma = (A \bar{\Lambda} A^{-1})^T - (A \bar{\Lambda} A^{-1})$ is (2.9) of section 2.1,that is the matrix identification of the vorticity vector. Taking $\hat{\tilde{\omega}} = -\bar{\Omega}$, $\hat{\tilde{\lambda}} = -\bar{\Lambda}$, $\hat{\xi} = -\Sigma$ we get the following expression obtained by Chandrasekhar [10]:

$$c = (a_2 a_3(2\tilde{\omega}_1 + \xi_1), a_1 a_3(2\tilde{\omega}_2 + \xi_2), a_1 a_2(2\tilde{\omega}_3 + \xi_3))$$

2.2.2 Momentum map

Define an action, Φ of $SO(3) \times SO(3)$ on $\mathcal{C} = GL^+(3)$ by:

$$\Phi : (SO(3) \times SO(3)) \times GL^+(3) \to GL^+(3)$$

$$\Phi((\lambda, \rho), Q) = \lambda Q \rho^T = \Phi_{(\lambda,\rho)}(Q)$$

This action induces an action on the phase space, $\mathcal{P} = T^*GL^+(3)$, called the lifted action on $\mathcal{P}$, which is defined on page 13 by equation (1.5). Note that the action on $\mathcal{C}$ respects the bipolar decomposition. The first $SO(3)$ factor acts by spatial rotation while the second gives rotation of the reference body.

An equivariant momentum map for the lifted action on $\mathcal{P}$, $J : T^*GL^+(3) \longrightarrow so(3)^* \times so(3)^*$, is given by

$$\langle J(\alpha), (\xi, \eta) \rangle = \ll \alpha, (\xi, \eta)_{GL^+(3)}(Q) \gg$$

where $(\xi, \eta)_{GL^+(3)}(Q)$ denotes the infinitesimal generator for the $SO(3) \times SO(3)$-action on $\mathcal{C}$.

The pairing $\ll \, , \, \gg$ is defined by

$$\ll \alpha, v \gg = \mathrm{tr}\,(\alpha^T v) \qquad \alpha \in T^*GL^+(3), v \in TGL^+(3)$$

and $\langle , \rangle$ is defined to be the restriction of the following pairing between $gl(3)^* \times gl(3)^*$ and $gl(3) \times gl(3)$

$$\langle (\alpha, \beta), (\xi, \eta) \rangle = \langle \alpha, \xi \rangle + \langle \beta, \eta \rangle = \mathrm{tr}\,(\alpha^T \xi + \beta^T \eta)$$

where

$$\langle \mu, \xi \rangle = \mathrm{tr}\,(\mu^T \xi) \qquad \mu \in gl(3)^*, \xi \in gl(3).$$

By definition, the infinitesimal generator for the $SO(3) \times SO(3)$-action on $GL^+(3)$ corresponding to $(\xi, \eta) \in so(3) \times so(3)$ is

$$(\xi, \eta)_{GL^+(3)}(Q) = \frac{d}{dt} \Phi_{(\exp t\xi, \exp t\eta)}(Q) \Big|_{t=0} = \xi Q + Q\eta^T = \xi Q - Q\eta$$

(note that $\xi = -\xi^T$ and $\eta = -\eta^T$).

With the above definitions in mind the momentum map is given by

$$\langle J(\alpha), (\xi, \eta) \rangle = \ll \alpha, \xi Q + Q\eta^T \gg = \mathrm{tr}\,(\alpha^T \xi Q + \alpha^T Q \eta^T) = \mathrm{tr}\,(Q\alpha^T \xi - \alpha^T Q \eta). \quad (2.19)$$

Note that in this expression the elements $Q\alpha^T$ and $\alpha^T Q$ are not skew-symmetric matrices and so do not give elements of $so(3)^* \times so(3)^*$. Now, for every matrix B we have the decomposition

$$B = \frac{1}{2}(B - B^T) + \frac{1}{2}(B + B^T) = B^{\mathcal{A}} + B^{\mathcal{S}},$$

into a skew-symmetric part, $B^{\mathcal{A}}$, and a symmetric part $B^{\mathcal{S}}$, and $\mathrm{tr}\,(ST) = 0$ for every symmetric matrix S and skew-symmetric matrix T. So, when we substitute for $Q\alpha^T$ and $\alpha^T Q$ by their skew-symmetric and symmetric parts in expression (2.19), it becomes equal to

$$\langle J(\alpha), (\xi, \eta) \rangle = \frac{1}{2}\mathrm{tr}\,\left[(Q\alpha^T - \alpha Q^T)\xi\right] + \frac{1}{2}\mathrm{tr}\,\left[(Q^T\alpha - \alpha^T Q)\eta\right]$$

which defines an element of $so(3)^* \times so(3)^*$. Hence the momentum map is given by

$$J(\alpha) = \frac{1}{2}(\alpha Q^T - Q\alpha^T, \alpha^T Q - Q^T \alpha) = (J_1, J_2) \quad (2.20)$$

As will be seen at the end of this section the momentum map components , J_1, J_2, can be identified with of the angular momentum and circulation.

For the generalized rigid body it is convenient to consider the so-called body and space coordinates, corresponding to the identification of the phase space $\mathcal{P} = T^*\Gamma$ with the product of the Lie group Γ with the dual of the Lie algebra of Γ, i.e $\Gamma \times \gamma^*$, (see example 1.1.2). The importance of these coordinates have been shown in first chapter, namely in the study of the momentum map which has particularly simple expressions in these coordinates .

In the affine rigid body case, $T^*GL^+(3)$ is identified with $GL^+(3) \times gl(3)^*$ by using one of the following isomorphisms $i_L, i_R : T^*_Q GL^+(3) \to GL^+(3) \times gl(3)^*$:

$$\begin{aligned} i_L(\alpha) &= (Q, \ll \alpha, T_e L_Q(\xi) \gg) = (Q, \mathrm{tr}\,(\alpha^T Q\xi)) = (Q, \langle Q^T\alpha, \xi\rangle) \\ i_R(\alpha) &= (Q, \ll \alpha, T_e R_Q(\xi) \gg) = (Q, \mathrm{tr}\,(\alpha^T \xi Q)) = (Q, \langle \alpha Q^T, \xi\rangle) \end{aligned} \tag{2.21}$$

for $\alpha \in T^*_Q GL^+(3)$. That is $i_L(\alpha) = (Q, Q^T\alpha)$ and $i_R(\alpha) = (Q, \alpha Q^T)$.

These isomorphisms in the affine rigid body do not give an expression for the momentum map only in terms of γ^* coordinates, as is the case when the action of Γ on Γ is the left or right action. However we do have:

$$J_L(Q,\beta) = J \circ i_L^{-1}(Q,\beta) = J(Q, Q^{-T}\beta) = \frac{1}{2}(Q^{-T}\beta Q^T - Q\beta^T Q^{-1}, \beta^T - \beta)$$

$$J_R(Q,\beta) = J \circ i_R^{-1}(Q,\beta) = J(Q, \beta Q^{-T}) = \frac{1}{2}(\beta - \beta^T, Q^{-1}\beta^T Q - Q^T\beta Q^{-T})$$

for $(Q,\beta) \in GL^+(3) \times gl(3)^*$ and J_R, J_L denoting the momentum map when the phase space identification with $GL^+(3) \times gl(3)^*$ is done by using respectively the isomorphisms i_R, i_L.

The momentum map for the lift of a given action on C to T^*C is always equivariant. That is, for the affine rigid body:

$$J\left(\Phi^{T^*}_{(\lambda,\rho)}(Q,\beta)\right) = \overline{Ad^*}_{(\lambda^T,\rho^T)} J(Q,\beta) \tag{2.22}$$

where Φ^{T^*} and $\overline{Ad^*}$ denote respectively the $SO(3) \times SO(3)$ -action on the phase space and the coadjoint action on $so(3)^* \times so(3)^*$.

We will use this equivariance property of the momentum map to compute explicitly the group action on the phase space in $GL^+(3) \times gl(3)^*$ coordinates.

Let us start with the explicit computation of the coadjoint action $\overline{Ad^*}$. This action is given by restriction of the usual coadjoint action of $GL^+(3) \times GL^+(3)$on $gl(3)^* \times gl(3)^*$. By definition the coadjoint action of $GL^+(3)$ on $gl(3)^*$ is:

$$\langle Ad^*_S\alpha, \xi\rangle = \langle \alpha, Ad_S\xi\rangle$$

where $Ad_S\xi = T_e(R_{S^{-1}} \circ L_S)(\xi) = S\xi S^{-1}$ with S and ξ matrices in $GL^+(3)$ and $gl(3)$ respectively, and R, L denoting the right and left actions respectively.

So, by definition of $\langle,\rangle$, we have

$$\langle Ad^*_S\alpha, \xi\rangle = \langle \alpha, S\xi S^{-1}\rangle = \mathrm{tr}\,(S^{-1}\alpha^T S\xi) = \langle S^T\alpha S^{-T}, \xi\rangle$$

that is, $Ad^*_S\alpha = S^T\alpha S^{-T}$ where $S^{-T} = (S^{-1})^T$. Taking $\overline{Ad^*_{(R,S)}}(\xi,\eta) = (Ad^*_R\xi, Ad^*_S\eta)$ and restricting R, S to $SO(3)$ and ξ, η to $so(3)^*$ we have

$$\overline{Ad^*_{(R,S)}}(\xi,\eta) = (R^T\xi R^{-T}, S^T\eta S^{-T}) = (R^T\xi R, S^T\eta S)$$

since $R^{-T} = (R^{-1})^T = R$ for $R \in SO(3)$.

So the equivariance expression (2.22) and the expressions of J_L, J_R in $GL^+(3) \times gl(3)^*$ coordinates give for the $SO(3) \times SO(3)$-action on the phase space , $\Phi^{T^*_L}, \Phi^{T^*_R}$ (the indices L, R denote which of the isomorphims i_L or i_R is used):

$$\begin{aligned} J_L\left(\Phi^{T^*}_{(\lambda,\rho)}(Q,\beta)\right) &= \overline{Ad^*_{(\lambda^T,\rho^T)}} J_L(Q,\beta) \\ &= \frac{1}{2}\left(\lambda\, Q^{-T}\beta Q^T\,\lambda^T - \lambda\, Q\beta^T Q^{-1}\,\lambda^T, \rho(\beta^T - \beta)\rho^T\right) \\ &= \frac{1}{2}\left((\lambda Q)^{-T}\beta(\lambda Q)^T - (\lambda Q)\beta^T(\lambda Q)^{-1}, \rho(\beta^T - \beta)\rho^T\right) \end{aligned}$$

that is $\Phi^{T^*_L}_{(\lambda,\rho)}(Q,\beta) = (\lambda Q\rho^T, \rho\beta\rho^T) = (\Phi_{(\lambda,\rho)}(Q), Ad^*_{\rho^T}\beta)$

$$\begin{aligned} J_R\left(\Phi^{T^*}_{(\lambda,\rho)}(Q,\beta)\right) &= \overline{Ad^*_{(\lambda^T,\rho^T)}} J_R(Q,\beta) \\ &= \frac{1}{2}\left(\lambda(\beta - \beta^T)\lambda^T, \rho\, Q^{-1}\beta^T Q\,\rho^T - \rho\, Q^T\beta Q^{-T}\rho^T\right) \\ &= \frac{1}{2}\left(\lambda(\beta - \beta^T)\lambda^T, (Q\rho^T)^{-1}\beta^T(Q\rho^T) - (Q\rho^T)^T\beta(Q\rho^T)^{-T}\right) \end{aligned}$$

So $\Phi^{T^*_R}_{(\lambda,\rho)}(Q,\beta) = (\lambda Q\rho^T, \lambda\beta\lambda^T) = (\Phi_{(\lambda,\rho)}(Q), Ad^*_{\lambda^T}\beta)$.

Note that the action on the phase space identified with $GL^+(3) \times gl(3)^*$ is just the given $SO(3) \times SO(3)$ action on $GL^+(3)$ and the usual coadjoint action on $gl(3)^*$.

The following proposition summarizes some of the main results we have just obtained.

Proposition 12 *The $SO(3) \times SO(3)$-action, Φ, on the configuration space $GL^+(3)$ induces a lifted action , Φ^{T^*}, on the phase space $T^*GL^+(3)$. The momentum map for the induced $SO(3) \times SO(3)$-action on the phase space is given by*

$$J(\alpha) = \frac{1}{2}(\alpha Q^T - Q\alpha^T, \alpha^T Q - Q^T\alpha) \tag{2.23}$$

for $\alpha \in T^*_Q GL^+(3)$. Identifying the phase space with $GL^+(3) \times gl(3)^*$ via one of the isomorphisms i_L, i_R, the $SO(3) \times SO(3)$- action on the phase space is given by the Φ-action on $GL^+(3)$ and the usual coadjoint action on $gl(3)^*$, that is

$$\Phi^{T^*_L}_{(\lambda,\rho)}(Q,\beta) = (\lambda Q\rho^T, \rho\beta\rho^T)$$

$$\Phi^{T^*_R}_{(\lambda,\rho)}(Q,\beta) = (\lambda Q\rho^T, \lambda\beta\lambda^T)$$

where the indices L, R indicates which of the isomorphisms i_L, i_R, respectively, is being used. The momentum map in $GL^+(3) \times gl(3)^*$ coordinates is given by:

$$\begin{aligned} J_L(Q,\beta) &= \frac{1}{2}(Q^{-T}\beta Q^T - Q\beta^T Q^{-1}, \beta^T - \beta) \\ J_R(Q,\beta) &= \frac{1}{2}(\beta - \beta^T, Q^{-1}\beta^T Q - Q^T\beta Q^{-T}). \end{aligned} \tag{2.24}$$

Next we will explore some properties of the momentum map J for the $SO(3) \times SO(3)$-action which are consequence of the presence of a symmetry other than $SO(3) \times SO(3)$ for the affine rigid body. Namely we shall see that there are $\mathbf{Z}_2$-actions on the phase space and on $SO(3) \times SO(3)$ such that J is also an equivariant momentum map for the semi-direct product action of $\mathbf{Z}_2$ by $SO(3) \times SO(3)$, i.e $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$.

Let $\mathbf{Z}_2 = \{1, \tau\}$ be the cyclic group of order two generated by τ. Consider the following $\mathbf{Z}_2$-actions on the configuration space $\mathcal{C} = GL^+(3)$ and on $SO(3) \times SO(3)$.

1) The $\mathbf{Z}_2$-action on $GL^+(3)$:

$$\Theta : \mathbf{Z}_2 \times GL^+(3) \to GL^+(3)$$

$$\Theta_\sigma(Q) = \begin{cases} Q & \text{for} \quad \sigma = 1 \\ Q^T & \text{for} \quad \sigma = \tau \end{cases}$$

2) The $\mathbf{Z}_2$-action on $SO(3) \times SO(3)$:

$$\sigma \cdot (x,y) = \begin{cases} (x,y) & \text{for} \quad \sigma = 1 \\ (y,x) & \text{for} \quad \sigma = \tau \end{cases}$$

The $\mathbf{Z}_2$-action on the phase space $T^*\mathcal{C}$ is the lift of the Θ-action. Denote this action by Θ^*_σ or by $\Theta^{K^*}_\sigma$ (with $K = \mathrm{L,R}$) when it is necessary to specify which isomorphism i_L or i_R is used to identify $T^*GL^+(3)$ with $GL^+(3) \times gl(3)^*$.

Proposition 13 *The lifted action of $\mathbf{Z}_2$ to $\mathcal{P}$, identified with $GL^+(3) \times gl(3)^*$, is the identity for $1 \in \mathbf{Z}_2$ and for the non trivial element:*

$$\Theta^{L^*}_\tau(Q,\beta) = (Q^T, Q\beta^T Q^{-1}) = (Q^T, Ad^*_{Q^T}\beta^T)$$

$$\Theta^{L^*}_\tau(Q,\beta) = (Q^T, Q^{-1}\beta^T Q) = (Q^T, Ad^*_{Q^{-T}}\beta^T)$$

for $(Q,\beta) \in GL^+(3) \times gl(3)^*$.

The momentum map J for the $(SO(3) \times SO(3))$-action is $\mathbf{Z}_2$-equivariant in the following sense:

$$J_K(\Theta^*_\sigma(Q,\beta)) = \sigma \cdot J_K(Q,\beta) \qquad \textit{for} \quad \mathrm{K = L, R} \quad \textit{and} \quad \sigma = 1, \tau \tag{2.25}$$

Proof:

By definition of lifted action we have:

$$\ll \Theta_\sigma^*(\alpha), v \gg = \ll T^*\Theta_{\sigma^{-1}}(\alpha), v \gg = \ll \alpha, T\Theta_{\sigma^{-1}}(v) \gg = \ll (T\Theta_{\sigma^{-1}})^*(\alpha), v \gg,$$

where $v \in T_{\Theta_\sigma(Q)}GL^+(3) = \begin{cases} T_Q GL^+(3) & \sigma = 1 \\ T_{Q^T} GL^+(3) & \sigma = \tau. \end{cases}$

That is $\Theta_\sigma^*(\alpha) = (T\Theta_{\sigma^{-1}})^*(\alpha)$. So as $\Theta_\sigma^{K^*}(Q,\beta) = (i_K \circ \Theta_\sigma^* \circ i_K^{-1})(Q,\beta)$ for $K = L, R$ then:

$$\Theta_\sigma^{L^*}(Q,\beta) = \begin{cases} i_L\left((T\Theta_{\sigma^{-1}})^*(Q^{-T}\beta)\right) = (Q, \ll (T\Theta_{\sigma^{-1}})^*(Q^{-T}\beta), T_eL_Q \gg & \sigma = 1 \\ i_L\left((T\Theta_{\sigma^{-1}})^*(Q^{-T}\beta)\right) = (Q^T, \ll (T\Theta_{\sigma^{-1}})^*(Q^{-T}\beta), T_eL_{Q^T} \gg & \sigma = \tau \end{cases}$$

where the last two right hand side equalities follow from the i_L definition and from the fact that the cotangent projection of $(T\Theta_{\sigma^{-1}})^*(Q^{-T}\beta)$ be Q^T and Q, respectively for $\sigma = \tau$ and $\sigma = 1$. For $\sigma = \tau$, we have:

$$\begin{aligned} \ll (T_{Q^T}\Theta_{\tau^{-1}})^*(Q^{-T}\beta), T_eL_{Q^T}(\xi) \gg &= \ll Q^{-T}\beta, \left(T_{Q^T}\Theta_{\tau^{-1}}^* \circ T_eL_{Q^T}\right)(\xi) \gg \\ &= \ll Q^{-T}\beta, T_{Q^T}\Theta_{\tau^{-1}}(Q^T\xi) \gg \\ &= \ll Q^{-T}\beta, \xi^T Q \gg = \operatorname{tr}(\beta^T Q^{-1}\xi^T Q) \\ &= \operatorname{tr}(\beta Ad_{Q^T}\xi) = \ll Ad_{Q^T}^*\beta^T, \xi \gg . \end{aligned}$$

So $\Theta_\tau^{L^*}(Q,\beta) = (Q^T, Q\beta^T Q^{-1}) = (Q^T, Ad_{Q^T}^*\beta^T)$.

For $\sigma = 1$ we have that $\Theta_\sigma^{L^*}(Q,\beta) = (Q,\beta)$.

Under the i_R identification we have, for $\sigma = \tau$,

$$\Theta_\sigma^{R^*}(Q,\beta) = i_R\left((T_{Q_T}\Theta_{\tau^{-1}})^*(\beta Q^{-T})\right) = \left(Q^T, \ll (T_{Q^T}\Theta_{\tau^{-1}})^*(\beta Q^{-T}), T_eR_{Q^T}(\xi) \gg\right)$$

The second element of the last pair is equal to

$$\ll \beta Q^{-T}, T_{Q^T}\Theta_{\tau^{-1}} \circ T_eR_{Q^T}(\xi) \gg = \ll \beta Q^{-T}, Q\xi^T \gg = \operatorname{tr}(\beta Ad_{Q^{-T}}\xi)$$

so $\Theta_\tau^{R^*}(Q,\beta) = (Q^T, Q^{-1}\beta^T Q) = (Q^T, Ad_{Q^T}^*\beta^T)$.

For $\sigma = 1$ the result is immediate.

We will show the equivariance (2.25) of J only for $K = L$, the other case is similar.

$$\begin{aligned} J_L(\Theta_\tau^{L^*}(Q,\beta)) &= J_L(Q^T, Q\beta^T Q^{-1}) \\ &= \frac{1}{2}\left(Q^{-1}Q\beta^T Q^{-1}Q - Q^T Q^{-T}\beta Q^T Q^{-T},\, Q^{-T}\beta Q^T - Q\beta^T Q^{-1}\right) \\ &= \frac{1}{2}\,\tau \cdot \left(Q^{-T}\beta Q - Q\beta^T Q^{-1},\, \beta^T - \beta\right) = \tau \cdot J_L(Q,\beta) \end{aligned}$$

and for $\sigma = 1$ is a straightforward consequence of the fact of Θ_1^* be the identity map. □

Remark: In [31], Chapter 8, Marsden discusses "discrete reduction" under two general assumptions about the symplectic actions of two Lie groups Σ and G, on a symplectic manifold P. The group Σ is compact and also acts on G by group homomorphisms, and hence also on the Lie algebra of G. His second assumption is that the momentum map, J, of the action of G, is Σ equivariant, i.e

$$J \circ \sigma_P = \sigma_{\mathcal{G}^*} \circ J \tag{2.26}$$

where σ_P is the Σ-action on P and $\sigma_{\mathcal{G}^*}$ is the derivative of the Σ-action on G at the identity. Equation (2.25) of the last proposition is (2.26) for $\Sigma = \mathbf{Z}_2$ and σ_P, and $\sigma_{\mathcal{G}^*}$ the Θ_σ and τ actions respectively. Furthermore the $\mathbf{Z}_2$ and $SO(3) \times SO(3)$ actions on the phase space, $GL^+(3) \times gl(3)^*$, and on the momentum space, $so(3)^* \times so(3)^*$, satisfy a compatibility condition which corresponds to Assumption 1 of Marsden [31]. This compatibility condition is (2.27) of the next proposition.

Proposition 14 i) *The* $\mathbf{Z}_2$ *and* $(SO(3) \times SO(3))$ *actions on* $\mathcal{P} = GL^+(3) \times gl(3)^*$, *respectively* Θ^*_σ *and* Φ^{T^*}, *satisfy, for* $\sigma = 1, \tau$:

$$\Phi^{T^*}_{(\lambda,\rho)}\left(\Theta^*_\sigma(Q,\beta)\right) = \Theta^*_\sigma\left(\Phi^{T^*}_{(\sigma\cdot(\lambda,\rho))}(Q,\beta)\right) \tag{2.27}$$

for both identifications of $\mathcal{P}$ *with* $GL^+(3) \times gl(3)^*$.

ii) *The* $\mathbf{Z}_2$ *and* $(SO(3) \times SO(3))$ *actions on* $so(3)^* \times so(3)^*$ *satisfy:*

$$\overline{Ad}^*_{(\lambda,\rho)}\left(\sigma \cdot (J_1, J_2)\right) = \sigma \cdot \left(\overline{Ad}^*_{(\sigma\cdot(\lambda,\rho))}(J_1, J_2)\right)$$

for $(J_1, J_2) \in so(3)^* \times so(3)^*$.

Proof: For i) and ii) respectively it is sufficient to prove:

$$\left(\Theta^*_\sigma \circ \Phi^{T^*}_{(\lambda,\rho)} \circ \Theta^*_{\sigma^{-1}}\right)(Q,\beta) = \Phi^{T^*}_{(\sigma\cdot(\lambda,\rho))}(Q,\beta)$$

$$\left(\sigma \circ \overline{Ad}^*_{(\lambda,\rho)} \circ \sigma^{-1}\right)(J_1, J_2) = \overline{Ad}^*_{\sigma\cdot(\lambda,\rho)}(J_1, J_2).$$

We will do it just for the case ii) since the other case is done in the same way. It is enough to prove only for the case $\sigma = \tau$ since for $\sigma = 1$ the result follows easily as a consequence of Θ^*_1 being the identity. So

$$\begin{aligned}\tau \cdot \left(\overline{Ad}^*_{(\lambda,\rho)}(J_2, J_1)\right) &= \tau \cdot \left(\lambda^T J_1 \lambda, \rho^T J_2 \rho\right) = (\rho^T J_2 \rho, \lambda^T J_1 \lambda) \\ &= \overline{Ad}^*_{(\rho,\lambda)}(J_2, J_1) = \overline{Ad}^*_{\tau\cdot(\lambda,\rho)}(J_1, J_2).\end{aligned}$$

□

As remarked in Marsden [31] the conditions (2.25) and (2.27) together enable us to show that the equivariant momentum map J for the $SO(3) \times SO(3)$-action is also an equivariant momentum map for the action of the semi-direct product, $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ with the following multiplication rule:

$$(\sigma_1, (\lambda_1, \rho_1))\,(\sigma_2, (\lambda_2, \rho_2)) = (\sigma_1\sigma_2, (\lambda_1, \rho_1)(\sigma_1 \cdot (\lambda_2, \rho_2))) \tag{2.28}$$

for $(\sigma_i, (\lambda_i, \rho_i)) \in \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ for $i, j = 1, 2$.

Theorem 5 *The $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ actions, ψ and ϕ on $so(3)^* \times so(3)^*$ and $GL^+(3) \times gl(3)^*$ are respectively given by*

$$\psi_{(\sigma,(\lambda,\rho))}(J_1, J_2) = \begin{cases} (\lambda^T J_1 \lambda, \rho^T J_2 \rho) & \text{for} \quad \sigma = 1 \\ \overline{Ad}^*_{(\lambda,\rho)}(\tau \cdot (J_1, J_2)) = (\lambda^T J_2 \lambda, \rho^T J_1 \rho) & \text{for} \quad \sigma = \tau \end{cases}$$

$$\phi^L_{(\sigma,(\lambda,\rho))}(Q, \beta) = \Phi^{T^*_L}_{(\lambda,\rho)}(\Theta^{L^*}_\sigma(Q,\beta)) = \begin{cases} (\lambda Q \rho^T, \rho \beta \rho^T) & \text{for} \quad \sigma = 1 \\ (\lambda Q^T \rho^T, \rho Q \beta^T Q^{-1} \rho^T) & \text{for} \quad \sigma = \tau \end{cases}$$

$$\phi^R_{(\sigma,(\lambda,\rho))}(Q, \beta) = \Phi^{T^*_R}_{(\lambda,\rho)}(\Theta^{R^*}_\sigma(Q,\beta)) = \begin{cases} (\lambda Q \rho^T, \rho \beta \rho^T) & \text{for} \quad \sigma = 1 \\ (\lambda Q^T \rho^T, \lambda Q^{-1} \beta^T Q \lambda^T) & \text{for} \quad \sigma = \tau. \end{cases}$$

The momentum map for the $SO(3) \times SO(3)$ action on $\mathcal{P}$, $J : GL^+(3) \times gl(3)^* \to so(3)^* \times so(3)^*$ is $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ equivariant in the following sense:

$$J_i\left(\phi^i_{(\sigma,(\lambda,\rho))}(Q,\beta)\right) = \psi_{(\tau,(\lambda^T,\rho^T))}(J_i(Q,\beta)) \tag{2.29}$$

for $i = L, R$ and $\sigma = 1, \tau$.

Proof

In order to prove that the expressions given above define an action, first note that the identity element of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$, i.e (1 , (I,I)) where I is the identity 3×3 matrix, maps an element of the semi-direct product into itsef.

Let $(\sigma_i, (\lambda_i, \rho_i))$ for $i = 1, 2$ be any two elements of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$. Then

$$\begin{aligned} \phi_{(\sigma_1,(\lambda_1,\rho_1))} \circ \phi_{(\sigma_2,(\lambda_2,\rho_2))} &= \overline{Ad}^*_{(\lambda_1,\lambda_2)} \circ \sigma_1 \circ \overline{Ad}^*_{(\lambda_2,\rho_2)} \circ \sigma_2 \\ &= \overline{Ad}^*_{(\lambda_1,\rho_1)} \circ \left(\sigma_1 \circ \overline{Ad}^*_{(\lambda_2,\rho_2)} \sigma_1^{-1}\right) \circ \sigma_1 \circ \sigma_2. \end{aligned}$$

By last proposition this becomes equal to

$$\overline{Ad}^*_{(\lambda_1,\rho_1)} \left(\overline{Ad}^*_{\sigma_1 \cdot (\lambda_2,\rho_2)} \circ \sigma_1 \cdot \sigma_2\right) = \overline{Ad}^*_{[(\lambda_1,\rho_1)(\sigma_1 \cdot (\lambda_2,\rho_2))]}(\sigma_1 \cdot \sigma_2) = \phi_{[\sigma_1\sigma_2,(\lambda_1,\rho_1)(\sigma_1 \cdot (\lambda_2,\rho_2))]},$$

which by definition (2.28) of the multiplication rule in the semi-direct product shows that ψ is an action.

For ϕ the proof follows in exactly the same way.

The equivariance of the momentum map J given in (2.29) is due to the $SO(3) \times SO(3)$ equivariance of J. That is

$$J_K\left(\Phi^{T^*_K}_{(\lambda,\rho)}(Q,\beta)\right) = \overline{Ad}^*_{(\lambda^T,\rho^T)} J_K(Q,\beta).$$

Then by proposition 13, equation (2.25),

$$\begin{aligned} J_K\left(\phi^K_{(\sigma,(\lambda,\rho))}(Q,\beta)\right) &= J_K\left(\Phi^{T^*_K}_{(\lambda,\rho)}(\Theta^{K^*}_\sigma(Q,\beta))\right) = \overline{Ad}^*_{(\lambda^T,\rho^T)} J_K(\Theta^{K^*}_\sigma(Q,\beta)) \\ &= \overline{Ad}^*_{(\lambda^T,\rho^T)} \sigma^{-1} \cdot J_K(Q,\beta) = \psi_{(\sigma,(\lambda^T,\rho^T))}\left(J_K(Q,\beta)\right). \end{aligned}$$

□

2.2.3 Momentum map and conserved quantities

In the Hamiltonian formulation the momentum P is related to $\dot{Q}$ by:

$$P = V\,E = \dot{Q}\,E$$

where

$$E = \int_{\mathcal{B}} (X \otimes X)\,\rho(X)\,dX$$

For a spherically symmetric reference body $\mathcal{B}$ we have $E = \frac{\mu}{2}\mathbf{I}$ and so $P = \frac{\mu}{2}\dot{Q}$.

The momentum map for the $SO(3) \times SO(3)$ action on $T^*GL^+(3)$ given by (2.20) is $J = (J_1\,,\,J_2)$ with

$$J_1 = \frac{1}{2}(P\,Q^T\;-\;Q\,P^T) \qquad J_2 = \frac{1}{2}(P^T\,Q\;-\;Q^T\,P).$$

Proposition 15 *The momentum map components J_1, J_2 are respectively the angular momentum and the Kelvin's circulation vector:*

$$J_1 = \frac{1}{2}\,\hat{\mathcal{A}} \qquad J_2 = -\frac{\mu}{4}\,\hat{c}.$$

Proof: By proposition 2.23 we have for momentum map $J = (J_1, J_2)$:

$$J_1 = \frac{1}{2}(P\,Q^T\;-\;Q\,P^T) = \frac{\mu}{4}(\dot{Q}\,Q^T\;-\;Q\,\dot{Q}^T)$$

$$J_2 = \frac{1}{2}(Q^T\,P -\;P^T\,Q) = -\frac{\mu}{4}(Q^T\,\dot{Q} - \dot{Q}^T\,Q)$$

By (2.18) we get:

$$\dot{Q}\,Q^T = \Omega\,Q\,Q^T - Q\,\Lambda\,Q^T + R^T\,\dot{A}\,A\,R$$

and

$$Q^T\,\dot{Q} = Q^T\,\Omega\,Q - Q^T\,Q\,\Lambda + S^T\,A\,\dot{A}\,S$$

Thus, from proposition 10, we have

$$J_1 = \frac{\mu}{4}\,\left[\Omega\,Q\,Q^T\;+\;Q\,Q^T\,\Omega\;-2\,Q\,\Lambda\,Q^T\right] = \frac{1}{2}\,\hat{\mathcal{A}},$$

and from proposition 11

$$J_2 = -\frac{\mu}{4}(2\,Q^T\,\Omega\,Q - (Q^T\,Q\,\Lambda + \Lambda\,Q^T\,Q)) = -\frac{\mu}{4}\,\hat{c}.$$

□

2.3 Dedekind's Theorem

Here we show how Dedekind's theorem is just a consequence of the symmetry $\mathbf{Z}_2$ and we explore the physical meaning of this symmetry.

Dedekind's theorem is concerned with the existence of ellipsoids of equilibrium of Dirichlet's problem which are adjoint (or congruent) to a given ellipsoid of equilibrium. According to Chandrasekhar [10], this theorem was proved by Dedekind only for ellipsoids congruent to the so-called Jacobi ellipsoids, which correspond to homogeneous rotating bodies with no internal motions, i.e vorticity equal zero, and with three different semi-axes lengths.

A Dedekind ellipsoid is an ellipsoid adjoint to a Jacobi ellipsoid and it is characterized by being stationary in space and having vorticity related to the angular velocity of its adjoint.

Let us analyse in more detail these two kinds of ellipsoids. From section 2.1, if X and x denote respectively a position in inertial frame and body frame coordinates, and $X(t) = Q(t)x(0)$, then the velocity in inertial and body frame coordinates is given by:

$$\dot{X} = F^T(\dot{x} - \Theta x) \qquad \dot{x} = u = \left(\dot{A}A^{-1} + A\Psi A^{-1}\right) x \tag{2.30}$$

where $Q = F^T A\, G$, $\Theta = \dot{F}\, F^T = -\hat{\theta}$, $\Psi = \dot{G}\, G^T = -\hat{\psi}$. Recall that this motion is one with angular velocity θ ($\hat{\theta} = -\Theta$) and vorticity ξ with $\hat{\xi} = (A\Psi A^{-1})^T - (A\Psi A^{-1})$.

Consider a Jacobi ellipsoid rotating with uniform angular velocity ω, say around the z-axis. Let $\omega = (0, 0, \omega)$ and A be a constant diagonal matrix having for entries the semi-axes lengths of the ellipsoid, a_1, a_2, a_3. Thus $Q = R^T A$ and $\Omega = -\hat{\omega}$. Taking in equations (2.30) $F = R$ and for G the identity matrix, we have:

- The vorticity of a Jacobi ellipsoid is zero.
- The motion of a Jacobi ellipsoid is given by:

$$\dot{U} = -\Omega\, U = \begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U$$

  where $U = RX\,R^T$ and $\dot{U} = R\,\dot{X}\,R^T$.

The adjoint configuration of the configuration Q is defined as being $Q^T = A\,R$. Hence applying (2.30) with $G = R$ and F the identity matrix we get:

- The Dedekind ellipsoid is stationary in space, that is with zero angular velocity, and has vorticity ξ where $\hat{\xi} = -\left[(A\,\Omega\,A^{-1})^T - (A\,\Omega\,A^{-1})\right]$
- The motion of the Dedekind ellipoid is given by:

$$u = A\,\Omega\,A^{-1}\,x = \begin{bmatrix} 0 & \omega a_1 & 0 \\ -\omega a_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{bmatrix}.$$

The analysis of the equations of motion governing the fluid, equation (2.11) (or equation (57) of chapter 4 of Chandrasekhar [10]) show that both Q and Q^T are solutions. This result can be generalized to any kind of ellipsoid of equilibrium under Dirichlet's assumptions.

Let us formulate more precisely Dedekind theorem as given in Chandrasekhar [10],§28, chapter 4. He states that if a motion determinated by $X(t)$ of the form (2.3), say $X(t) = Q(t)x_0$ with $x_0 = I$, is admissible under Dirichlet's conditions then the motion determined by Q^T is also admissible and the configurations Q and Q^T are called adjoint configurations.

Let us now see how Dedekind's theorem is a consequence of the $\mathbf{Z}_2$ symmetry. Recall that $\mathbf{Z}_2$ acts on the configuration space, $GL^+(3)$, by transposition and on $so(3)^* \times so(3)^*$ by interchanging the $so(3)^*$ factors. From the properties of symmetric dynamical systems, if we have an $SO(3) \times SO(3)$ group orbit for Dirichlet's problem then by applying the $\mathbf{Z}_2$ symmetry we get another invariant group orbit. The following results give the physical interpretation of this $\mathbf{Z}_2$ action.

Proposition 16 *A configuration Q and its transpose have corresponding momentum map components interchanged. That is:*

$$J_1(P_Q\,,\,Q) = J_2(P_{Q^T}\,,\,Q^T) \qquad J_2(P_Q\,,\,Q) = J_1(P_{Q^T}\,,\,Q^T)$$

Before proving this proposition let us establish the following lemma which relates the angular momenta and circulations of adjoint configurations.

Lemma 5 *Let $\hat{\mathcal{A}}_Q$ and $\hat{c}_Q$ be, respectively, the angular momentum and the circulation of a configuration Q.*

The angular momenta and circulations of the configurations Q and Q^T are related by the formulae:

$$\begin{aligned} \hat{\mathcal{A}}_{Q^T} &= -\frac{\mu}{2}\,\hat{c}_Q \\ \hat{c}_{Q^T} &= -\frac{2}{\mu}\,\hat{\mathcal{A}}_Q. \end{aligned}$$

Proof: As in section 2.2.1, let $Q = R^T A\, S$ and $\Omega = \dot{R}^T R\,,\ \Lambda = \dot{S}^T S$. If we consider a spherically symmetric reference body, propositions 10 and 11 give for the angular momentum and circulation of Q respectively:

$$\hat{\mathcal{A}}_Q = \frac{\mu}{2}\left\{\Omega\,(QQ^T) + (QQ^T)\,\Omega - 2Q\,\Lambda\,Q^{-T}\right\}$$

$$\hat{c}_Q = (Q^T\,\dot{Q} - \dot{Q}^T\,Q) = 2Q^T\,\Omega\,Q - \left[Q^T\,Q\,\Lambda + \Lambda\,Q^T\,Q\right].$$

The definitions, of the angular momentum and circulation for Q^T are respectively $\hat{\mathcal{A}}_{Q^T} = \int_B (Q^T X) \times (\dot{Q}^T X)\,dX$ and $c_{Q^T} = (Q\,\dot{Q}^T - \dot{Q}\,Q^T)$.

Following the proof of proposition 10 we get easily:

$$\hat{\mathcal{A}}_{Q^T} = \frac{\mu}{2}\left\{\Lambda\,(Q^T Q) + (Q^T Q)\,\Lambda - 2Q^T\,\Lambda\,Q^{-1}\right\}.$$

Calculating $\dot{Q}^T$ we have:

$$\hat{c}_{Q^T} = (Q^T\,\dot{Q} - \dot{Q}^T\,Q) = 2Q\,\Lambda\,Q^T - \left[Q\,Q^T\,\Omega + \Omega\,Q\,Q^T\right].$$

Comparing the respective expressions we have the result.

□

Proof (proposition 16) : From proposition 15 we have

$$J_1(P_Q, Q) = \frac{1}{2}\,\hat{\mathcal{A}}_Q \qquad J_2(P_Q, Q) = -\frac{\mu}{4}\,\hat{c}_Q.$$

Thus, by last lemma, we get:

$$J_1(P_Q, Q) = \frac{1}{2}\,\hat{\mathcal{A}}_Q = -\frac{\mu}{4}\,\hat{c}_{Q^T} = J_2(P_{Q^T}, Q^T)$$

$$J_2(P_Q, Q) = -\frac{\mu}{4}\,\hat{c}_Q = \frac{1}{2}\,\hat{\mathcal{A}}_{Q^T} = J_1(P_{Q^T}, Q^T).$$

□

Remarks:

a) As we can see from what has been said for Dedekind and Jacobi ellipsoids, adjoint ellipsoids are physically very different from each other. Indeed, from the last proposition we have that the motions of adjoint configurations have their angular momenta and circulations interchanged.

b) There exist configurations which are their own transposes. These kind of configurations are called "self-adjoint". From lemma 5 the self-adjoint configurations have angular momentum, $\hat{\mathcal{A}}_Q$, equal to $-\frac{\mu}{2}\,\hat{c}_Q$, or

$$J_1(P_Q, Q) = J_2(P_{Q^T}, Q^T) = J_2(P_Q, Q).$$

2.4 Isotropy Subgroups

This section will be devoted to finding the isotropy lattices for the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ actions on the phase space $\mathcal{P} = GL^+(3) \times gl(3)^*$ and on the momentum space $so(3)^* \times so(3)^*$. Let be $\Gamma = \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ in what follows.

The isotropy lattice of the action of a group Γ on a set is the set of conjugacy classes $[\Sigma]$ of isotropy subgroups Σ of Γ partially ordered by $[\Sigma] < [T]$, where $[\Sigma] < [T]$ if and only if there is γ such that $\gamma\Sigma\gamma^{-1} \subset T$.

By the definition of the isotropy subgroup of a point and by the expressions for the actions given in theorem 5 we have that $\Sigma \subset \Gamma$ is an isotropy subgroup for the Γ-action on $so(3)^* \times so(3)^*$ if and only if

$$\Sigma = \{(\sigma,(\lambda,\rho)) \in \Gamma \ : \psi_{(\sigma,(\lambda,\rho))}(\alpha,\mu) = (\alpha,\mu)\}$$

and for the Γ action on $GL^+(3) \times gl(3)^*$ if and only if

$$\Sigma = \{(\sigma,(\lambda,\rho)) \in \Gamma \ : \phi^K_{(\sigma,(\lambda,\rho))}(Q,\beta) = (Q,\beta) \quad \text{for} \quad \mathrm{K} = \mathrm{L},\mathrm{R}\}.$$

That is:

a) For $(Q,\beta) \in GL^+(3) \times gl(3)^*$

$$\Sigma^L_{(Q,\beta)} = \{(\sigma,(\lambda,\rho)) \in \Gamma \ : (\lambda Q\rho^T, \rho\beta\rho^T) = (Q,\beta) \quad \text{for} \quad \sigma = 1 \quad \text{and}$$
$$(\lambda Q^T\rho^T, \rho Q\beta^T Q^{-1}\rho^T) = (Q,\beta) \quad \text{for} \quad \sigma = \tau\}$$

$$\Sigma^R_{(Q,\beta)} = \{(\sigma,(\lambda,\rho)) \in \Gamma \ : (\lambda Q\rho^T, \lambda\beta\lambda^T) = (Q,\beta) \quad \text{for} \quad \sigma = 1 \quad \text{and}$$
$$(\lambda Q^T\rho^T, \lambda Q\beta^T Q^{-1}\lambda^T) = (Q,\beta) \quad \text{for} \quad \sigma = \tau\}$$

where the indexes L, R denote the left and the right identifications of $T^*GL^+(3)$ with $GL^+(3) \times gl(3)^*$.

b) For $(\alpha,\mu) \in so(3)^* \times so(3)^*$

$$\Sigma_{(\alpha,\mu)} = \{(\sigma,(\lambda,\rho)) \in \Gamma \ : (\lambda^T\mu\lambda, \rho^T\alpha\rho) = (\alpha,\mu) \quad \text{for} \quad \sigma = \tau \quad \text{and}$$
$$(\lambda^T\alpha\lambda, \rho^T\mu\rho) = (\alpha,\mu) \quad \text{for} \quad \sigma = 1\}$$

The conjugation action on $\Gamma = \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ follows from the definition (2.28) of multiplication in $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$. The inverse of an element $(\sigma,(\lambda,\rho)) \in \Gamma$ is $(\sigma, \sigma \cdot (\lambda^T,\rho^T))$ since $\sigma^{-1} = \sigma$ for any $\sigma \in \mathbf{Z}_2$. Thus a subgroup $\Sigma_1 \subset \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ is conjugate to Σ_2 if and only if there is a $(t,(\mu,\theta)) \in \Gamma$ such that for all $(\sigma,(\lambda,\rho)) \in \Sigma_1$

$$(t,(\mu,\theta))\,(\sigma,(\lambda,\rho))\,(t, t\cdot(\mu^T,\theta^T)) = \Sigma_2$$

i.e

$$\Big(\sigma,(\mu,\theta)\ \ (t\cdot(\lambda,\rho))\ \ (\sigma\cdot(\mu^T,\theta^T))\Big) = \Sigma_2. \tag{2.31}$$

As a straightforward consequence of (2.28) we have the following lemma.

Lemma 6 *Every subgroup of* $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ *of the form* $\mathbf{1} \times_s (A \times B)$ *is conjugate to* $\mathbf{1} \times_s (B \times A)$.

Proof: From (2.31) with $(t,(\mu,\theta)) = (\tau,(\mathrm{I},\mathrm{I}))$ the result follows.

□

Hereafter whenever we study the Γ-action on the phase space we only consider the left identification of $T^*GL^+(3)$ with $GL^+(3) \times gl(3)^*$ or in other words only isotropy subgroups of the form $\Sigma^L_{(Q,\beta)}$.

2.4.1 Isotropy subgroups for the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ action on $so(3)^* \times so(3)^*$

The isotropy subgroups of the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$-action on $so(3)^* \times so(3)^*$ are closely related to the isotropy subgroups of the usual $SO(3)$-action on $so(3)$ (or $so(3)^*$) by conjugation, i.e $SO(3) \times so(3) \ni (\lambda, \alpha) \mapsto \lambda\alpha\lambda^T$.

The representation of $SO(3)$ on $so(3)^*$, where $SO(3)$ acts by conjugation (i.e the coadjoint action), is isomorphic to the representation of $SO(3)$ on the space V_1 of the spherical harmonics of degree 1, as we will show next. This is described in Golubitsky *et al.* [16] chapter XIII.

The space $so(3)^*$ can be identify with $\mathbf{R}^3$ or the 3-dimensional space V_1 of the homogeneous polynomials $p : \mathbf{R}^3 \to \mathbf{R}$ of degree 1, and $SO(3)$ acts on V_1 by $\gamma \cdot p(x) = p(\gamma^T x)$.

Proposition 17 *The representation of $SO(3)$ on V_1 is isomorphic to the representation of $SO(3)$ on $so(3)^*$, where $SO(3)$ acts on $so(3)^*$ by the coadjoint action Ad^*.*

Proof: Any polynomial $p \in V_1$ can be represented as the scalar product of two vectors $a, x \in \mathbf{R}^3$, that is $p(x) = (a\,,\,x) = p_a(x)$ for some a. Identify $\mathbf{R}^3$ with $so(3)$ via $l : a \in \mathbf{R}^3 \mapsto A \in so(3)$. Let $l_1 : V_1 \to so(3)$ be $l_1(p(x)) = l_1(p_a(x)) = l(a) = A$. Easy calculations show that, for $\gamma \in SO(3)$,

$$\gamma \cdot p_a(x) = p_a(\gamma^T x) = (a\,,\,\gamma^T x) = (\gamma a\,,\,x) = p_{\gamma a}(x).$$

As $l^{-1}(Ad_\gamma l(a)) = \gamma a$ with $\gamma \in SO(3)$ and $Ad_\gamma l(a) = \gamma A \gamma^T$, then $Ad_\gamma(l_1(p_a(x))) = Ad_\gamma l(a) = l(\gamma a) = l_1(p_a(\gamma^T x)) = l_1(\gamma \cdot p(x))$ which is the same of saying that the following diagram commutes

$$\begin{tikzcd} V_1 \arrow[r, dash, "l_1"] \arrow[d, "\gamma"'] & so(3) \arrow[d, dash, "Ad_\gamma"] \\ V_1 \arrow[r, "l_1"'] & so(3) \end{tikzcd}$$

Identifying $so(3)$ with $so(3)^*$, we get the result.

$\square$

By Golubitsky *et al.* [16] chapter XIII all the isotropy subgroups, except $SO(3)$, of the $SO(3)$ 3-dimensional representation are conjugate to $SO(2)$. The isotropy lattice of the 3-dimensional representation of $SO(3)$ is as shown in figure 2.1.

Every $SO(3)$ orbit in $so(3)^*$ with isotropy conjugate to $SO(2)$ has a representative point of the form:

$$A = \begin{pmatrix} 0 & -a & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for} \quad a \in \mathbf{R}. \tag{2.32}$$

$$\begin{array}{c} SO(3) \\ \uparrow \\ SO(2) \end{array}$$

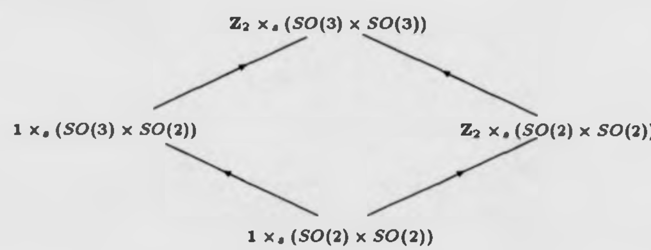
Figure 2.1: Isotropy lattice of $SO(3)$ on V_1.

The $SO(3) \times SO(3)$-action on $so(3)^* \times so(3)^*$ is $Ad^*_{(\lambda,\rho)}(J_1, J_2) = (\lambda^T J_1 \lambda, \rho^T J_2 \rho)$,. So the orbit representatives and the corresponding isotropy subgroups are listed in table 2.1.

Orbit Representative	Isotropy Subgroup
$(0,0)$	$SO(3) \times SO(3)$
$(A,0)$	$SO(2) \times SO(3)$
$(0,A)$	$SO(3) \times SO(2)$
(A,B)	$SO(2) \times SO(2)$

Table 2.1: Isotropy subgroups and orbit representatives for the action of $SO(3) \times SO(3)$ on $so(3)^* \times so(3)^*$. The matrices A and B have the form given in (2.32).

Proposition 18 *The isotropy lattice for the action of* $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ *on* $so(3)^* \times so(3)^*$ *is given by*



Proof: Let $\Sigma_1 \times \Sigma_2 \in SO(3) \times SO(3)$ be an isotropy subgroup for the $SO(3) \times SO(3)$-action on $so(3)^* \times so(3)^*$.

Let us show that the only isotropy subgroups for the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$-action on $so(3)^* \times so(3)^*$ which are not of the form $\mathbf{1} \times_s (\Sigma_1 \times \Sigma_2)$ are $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ and $\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$.

From the definitions of the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$-action and of isotropy subgroups we have, for $\sigma = 1 \in \mathbf{Z}_2$, that $(1\,,\,(\lambda, \rho))$ belongs to the isotropy subgroup iff

$$\overline{Ad}^*_{(\lambda^T,\rho^T)}(J_1, J_2) = (J_1, J_2), \tag{2.33}$$

that is $(\lambda^T, \rho^T) \in \Sigma_1 \times \Sigma_2$.

For $\sigma = \tau$, then $(\tau, (\lambda, \rho))$ fixes (J_1, J_2) iff

$$\overline{Ad}^*_{(\lambda^T, \rho^T)}(J_1, J_2) = (J_2, J_1)$$

holds. By the analysis of the orbit reprentatives of the $SO(3) \times SO(3)$-action on $so(3)^* \times so(3)^*$ the only subgroups are $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ and $\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$ which fix respectively $(0,0)$ and (J, J).

The isotropy lattice now follows from the fact that every subgroup of the form $\mathbf{1} \times_s (\Sigma_1 \times \Sigma_2)$ is conjugate to $\mathbf{1} \times_s (\Sigma_2 \times \Sigma_1)$.

□

Remarks:

a) The list of the orbit reprentatives for the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ action on $so(3)^* \times so(3)^*$ is:

Orbit Representative	$\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ Isotropy Subgroup
$(0,0)$	$\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$
$(0, A)$	$\mathbf{1} \times_s (SO(3) \times SO(2))$
$(A, \pm A)$	$\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$
(A, B)	$\mathbf{1} \times_s (SO(2) \times SO(2))$

where A and B are matrices of the form (2.32). Note that both (A, A) and $(A, -A)$ are orbit representatives for the isotropy subgroup $\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$ since an element $\lambda \in SO(3)$ fixes an element $A \in so(3)^*$ if and only if it fixes $-A$. So $(\lambda J_1 \lambda^T, \rho(-J_1)\lambda^T) = (-J_1, J_1)$ for $(\lambda, \rho) \in SO(3) \times SO(3)$.

b) Let us call J_1 angular momentum and J_2 circulation. The fixed point sets of the isotropy subgroups, up to conjugacy, can be interpreted with some abuse of language as

$\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$	angular momentum = circulation =0
$\mathbf{1} \times_s SO(3) \times SO(2)$	angular momentum or circulation equal to zero or A
$\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$	angular momentum = $\pm$ circulation = $\pm A$
$\mathbf{1} \times_s (SO(2) \times SO(2))$	angular momentum, circulation of the form A .

2.4.2 Isotropy subgroups of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on $GL^+(3) \times gl(3)^*$ and fixed point sets

The isotropy subgroups for the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$-action on $GL^+(3) \times gl(3)^*$ are closely related to the $SO(3)$ actions on $GL^+(3)$ and $gl(3)^*$, where the $SO(3)$- action on $gl(3)^*$ is given by conjugation, as well as to the isotropy subgroups of an action of $\mathbf{Z}_2^\tau \times SO(3)$ on $gl(3)^*$ which will be defined in (2.34).

Let us start with the analysis of the lattice of isotropy subgroups for the $SO(3)$ action on $gl(3)^*$ given by conjugation $(\lambda \cdot \beta = \lambda \beta \lambda^T)$. The set of 3×3 matrices, $gl(3)^*$, can be identify with $\mathbf{R}^9 \cong V_0 \oplus V_1 \oplus V_2$, where V_l is the $2l+1$ dimensional space of spherical harmonics of degree l. Furthermore we have that $SO(3)$ acts on each of the $SO(3)$ invariant spaces V_l by conjugation and

- V_0 is isomorphic to the space of scalar multiples of the identity matrices.
- V_1 is isomorphic to the space of skew symmetric matrices.
- V_2 is isomorphic to the space of symmetric matrices of trace zero.

Note that the action of $SO(3)$ on V_2 follows from the identification of this space with that of homogeneous polynomials p of degree 2 in $\mathbf{R}^3$. If $p(x) = x^T A x$, where A is a symmetric matrix, then $p(\gamma^T x) = x^T \gamma A \gamma^T x$.

The lattice of isotropy subgroups of V_1 is given in figure 2.1 while the isotropy lattice of V_2 is shown in figure 2.2 (see Golubitsky *et al.* [16] pg.232).

$$\begin{array}{c} SO(3) \\ \uparrow \\ O(2) \\ \uparrow \\ D_2 \end{array}$$

Figure 2.2: Isotropy subgroups of $SO(3)$ on V_2.

As the $SO(3)$ action on V_0 is trivial, the isotropy subgroups of the $SO(3)$ action on $gl(3)^*$ are the intersections of isotropy subgroups of V_1 and V_2. The conjugacy classes of these intersections give the lattice of isotropy subgroups of figure 2.3 and the fixed point spaces of representative isotropy subgroups are given in table 2.2.

In order to find the isotropy subgroups of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on $gl(3)^*$ we will study the isotropy subgroups of the action of $\mathbf{Z}_2^\tau \times SO(3)$ on $gl(3)^*$, where $\mathbf{Z}_2^\tau = \{1, \tau\}$ with $\tau^2 = 1$, given by:

$$\begin{cases} (1,\rho)\cdot\beta &= \rho\beta\rho^T \\ \\ (\tau,\rho)\cdot\beta &= \rho\beta^T\rho^T \end{cases} \tag{2.34}$$

Let $\Sigma \subset \mathbf{Z}_2^\tau \times SO(3)$ be an isotropy subgroup for the action defined by (2.34). Denote by Σ_1 and Σ_2 the intersection of Σ respectively with $\{1\} \times SO(3)$ and $\{\tau\} \times SO(3)$ and by π the projection $\pi : \mathbf{Z}_2^\tau \times SO(3) \to SO(3)$.

Note that $\pi(\Sigma_1)$ is an isotropy subgroup of $SO(3)$ on $gl(3)^*$ fixing an element β of the form listed in the second column of table 2.2. If this β is symmetric ($\beta = \beta^T$) then $\pi(\Sigma_1) = \pi(\Sigma_2)$ and $\Sigma = \mathbf{Z}_2^\tau \times \pi(\Sigma_2)$.

When $\pi(\Sigma_1)$ fixes a $\beta \neq \beta^T$ and $\pi(\Sigma_2)$ is non-empty then we claim that $\pi(\Sigma_1)$ is a subgroup of index 2 in $\pi(\Sigma)$, where the index of a subgroup $\pi(\Sigma_1)$ in $\pi(\Sigma)$ is the number of cosets in the quotient $\dfrac{\pi(\Sigma)}{\pi(\Sigma_1)}$. Let us prove this claim:

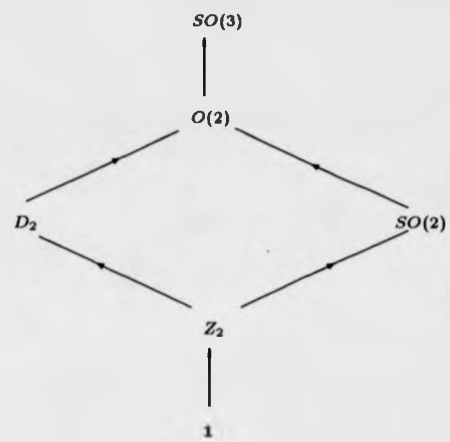


Figure 2.3: Isotropy subgroups of $SO(3)$ on $gl(3)^*$.

By definition of Σ_2 any $\rho_1, \rho_2 \in \pi(\Sigma) \sim \pi(\Sigma_1)$ correspond to elements $(\tau, \rho_i) \in \Sigma_2$ (i=1,2). As ρ_1^{-1} also belong to $\pi(\Sigma) \sim \pi(\Sigma_1)$ then

$$(\tau, \rho_1^{-1})(\tau, \rho_2) = (1, \rho_1^{-1}\rho_2) \in \pi(\Sigma_1)$$

that is $\rho_2 \in \rho_1\pi(\Sigma_1)$, which is the same of saying that ρ_1 and ρ_2 are in the same coset of $\pi(\Sigma_1)$ in $\pi(\Sigma)$.

Note that if $\beta \neq \beta^T$ then $\pi(\Sigma)$ is a subgroup of $SO(3)$ while $\pi(\Sigma_2)$ is not, since the identity does not belong to it.

The possible $\pi(\Sigma_1)$'s are the isotropy subgroups of the $SO(3)$ action on $gl(3)^*$, i.e the isotropy subgroups given in figure 2.3 and their conjugates.

We will divide the study of the isotropy subgroups of the $\mathbf{Z}_2^\tau \times SO(3)$ action by considering the three types of subgroups:

(1) $\mathbf{Z}_2^\tau \times \Sigma$.

(2) $\tilde{\Sigma}$.

(3) $\mathbf{1} \times \Sigma$.

In each of these cases Σ is an isotropy subgroup of $SO(3)$. In case (2), $\pi(\tilde{\Sigma}) = \Sigma$ and $\tilde{\Sigma} \cap (\mathbf{1} \times SO(3))$ is a subgroup of index 2 in Σ.

Case (1):

When $\pi(\Sigma_1)$ is equal to $SO(3), O(2)$ or D_2 the element β fixed by each of them is symmetric and so the isotropy subgroup Σ is, respectively:

$$\mathbf{Z}_2^\tau \times SO(3) \qquad \mathbf{Z}_2^\tau \times O(2) \qquad \mathbf{Z}_2^\tau \times D_2.$$

Isotropy Subgroup	Fixed point space
$SO(3)$	kI
$O(2)$	$\begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix}$
$SO(2)$	$\begin{pmatrix} a & -d & \\ d & a & \\ & & b \end{pmatrix}$
D_2	$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$
$\mathbf{Z}_2$	$\begin{pmatrix} a & d & \\ g & b & \\ & & c \end{pmatrix}$
$\mathbf{1}$	$gl(3)^*$

Table 2.2: Fixed point sets of representative isotropy subgroups of $SO(3)$ on $gl(3)^*$.

If $\pi(\Sigma_1)$ is $\mathbf{Z}_2$ or $\mathbf{1}$ then with $\pi(\Sigma_2) = \{1\}$ there exist isotropy subgroups, say $\mathbf{Z}_2^\tau \times \mathbf{Z}_2$, $\mathbf{Z}_2^\tau$, fixing respectively:

$$\begin{pmatrix} a & d & \\ d & b & \\ & & c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}.$$

Case (2):

The isotropy subgroups of $SO(3)$ fixing non-symmetric β's are $SO(2)\,, \mathbf{Z}_2\,, \mathbf{1}$. When $\pi(\Sigma_1)$ is equal to $SO(2), \mathbf{Z}_2$ or $\mathbf{1}$ and $\pi(\Sigma_2)$ non-empty, then $\pi(\Sigma_1)$ is of index two in $\pi(\Sigma)$. We can enumerate, up to conjugacy, pairs of subgroups $\pi(\Sigma_1)$, $\pi(\Sigma)$, with $\pi(\Sigma_1) \subset \pi(\Sigma)$, such that $\pi(\Sigma_1)$ is of index 2 in $\pi(\Sigma)$. These are:

a) $SO(2) \subset O(2)$

b) $\mathbf{Z}_2 \subset \mathbf{Z}_4$

c) $\mathbf{Z}_2 \subset D_2$

d) $\mathbf{1} \subset \mathbf{Z}_2$

Thus for β belonging to the fixed point set of $SO(2)\,, \mathbf{Z}_2\,, \mathbf{1}$ we will look for elements $\rho \in \pi(\Sigma_2)$ such that $\rho^2 = 1$, $\rho \in \pi(\Sigma) \sim \pi(\Sigma_1)$ and $\rho\beta^T\rho = \beta$. These conditions and the above inclusion relations are enough to determine Σ.

If $\pi(\Sigma_1) = SO(2)$ then choosing ρ:

$$\rho = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \tag{2.35}$$

then $(\tau\,,\,\rho)\cdot\beta = \rho\beta^T\rho = \beta$, where β belongs to the fixed point set of $SO(2)$ of table 2.2. The projection of the isotropy subgroup generated by $\pi(\Sigma_1) = SO(2)$ and $(\tau\ \rho)$ (ρ as in (2.35)) is $O(2)$. We denote this isotropy subgroup of $\mathbf{Z}_2 \times SO(3)$ by $\widetilde{O(2)}$.

For $\pi(\Sigma_1) = \mathbf{Z}_2$ we can choose elements $\rho \in \pi(\Sigma_2)$ lying in one of the sets:

- $\mathbf{Z}_4 \sim \mathbf{Z}_2$.
- $\mathbf{D}_2 \sim \mathbf{Z}_2$.

In the first case, $\rho\ \in \mathbf{Z}_4 \sim \mathbf{Z}_2$, it is not hard to see that the fixed point set is the same as the one of $\mathbf{Z}_2^\tau \times O(2)$, so the group Σ is not an isotropy subgroup.

Taking now $\rho\ \in \mathbf{D}_2 \sim \mathbf{Z}_2$ of the form (2.35) we get for Σ an isotropy subgroup, with $\pi(\Sigma_1) = \mathbf{Z}_2$, such that $\pi(\Sigma)$ is isomorphic to D_2 and we denote it by $\widetilde{D_2}$. It is not hard to see that this group fixes an element β of the form

$$\begin{pmatrix} a & -d & \\ d & b & \\ & & c \end{pmatrix}$$

where we have choosen the nontrivial element of $\mathbf{Z}_2$ to be $\mathrm{diag}(-1,-1,1)$.

Consider now the case $\pi(\Sigma_1) = \mathbf{1}$:

For $\rho \in \mathbf{Z}_2 \sim \mathbf{1}$ we get $\pi(\Sigma)$ isomorphic to $\mathbf{Z}_2$ and we denote Σ by $\widetilde{\mathbf{Z}_2}$. For the element ρ we can take

$$\rho = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

For this representative the fixed point space consists of matrices of the form:

$$\begin{pmatrix} a & d & -e \\ d & b & -f \\ e & f & c \end{pmatrix}.$$

Case (3): The only isotropy subgroups of $\mathbf{Z}_2 \times SO(3)$ of the form $\mathbf{1} \times \Sigma$ are $\mathbf{1} \times \mathbf{Z}_2 = \mathbf{Z}_2$ and $\mathbf{1} \times \mathbf{1} = \mathbf{1}$.

Putting everything together, we get figure 2.4 for the isotropy lattice of $\mathbf{Z}_2^\tau \times SO(3)$ on $gl(3)^*$ and table 2.3 for fixed point sets of representative isotropy subgroups .

Isotropy subgroup	Fixed point set	Dimension of fixed point set
$\mathbf{Z}_2^\tau \times SO(3)$	aI	1
$\mathbf{Z}_2^\tau \times O(2)$	$\begin{pmatrix} a & & \\ & a & \\ & & b \end{pmatrix}$	2
$\mathbf{Z}_2^\tau \times D_2$	$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$	3
$\widetilde{O(2)}$	$\begin{pmatrix} a & -d & \\ d & a & \\ & & c \end{pmatrix}$	3
$\mathbf{Z}_2^\tau \times \mathbf{Z}_2$	$\begin{pmatrix} a & d & \\ d & b & \\ & & c \end{pmatrix}$	4
$\widetilde{D_2}$	$\begin{pmatrix} a & -d & \\ d & b & \\ & & c \end{pmatrix}$	4
$\mathbf{Z}_2^\tau$	$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}$	6
$\mathbf{1} \times \mathbf{Z}_2 \simeq \mathbf{Z}_2$	$\begin{pmatrix} a & d & \\ e & b & \\ & & c \end{pmatrix}$	5
$\widetilde{\mathbf{Z}_2}$	$\begin{pmatrix} a & d & -e \\ d & b & -f \\ e & f & c \end{pmatrix}$	6
$\mathbf{1}$	$gl(3)^*$	9

Table 2.3: Fixed point spaces of representative isotropy subgroups of $\mathbf{Z}_2^\tau \times SO(3)$ on $gl(3)^*$.

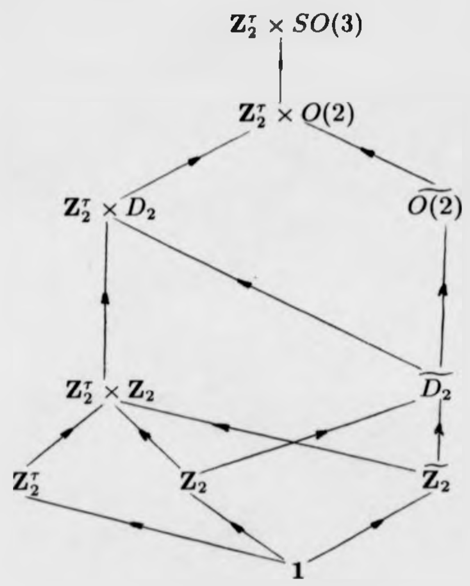


Figure 2.4: Isotropy subgroups of $\mathbf{Z}_2^\tau \times SO(3)$ on $gl(3)^*$.

Finally let us consider the lattice of isotropy subgroups of the action of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on $GL^+(3) \times gl(3)^*$. This action is given by

$$\begin{aligned} (1, \lambda, \rho) \cdot (Q, \beta) &= (\lambda\, Q\, \rho^T, \rho\, \beta\, \rho^T) \\ (\tau, \lambda, \rho) \cdot (Q, \beta) &= (\lambda\, Q^T\, \rho^T, \rho\, Q\, \beta^T\, Q^{-1}\, \rho^T) \end{aligned} \tag{2.36}$$

Every point (Q, β) is contained in the $SO(3) \times SO(3)$ orbit of a point with Q diagonal. It is therefore sufficient to consider only the case when Q is diagonal. We may also assume that all entries of Q are positive.

Note that if Q is diagonal $\lambda Q^T \rho^T = Q \iff \lambda Q \rho^T = Q$.

Lemma 7 *If $\lambda Q \rho^T = Q$ and Q is diagonal then $\lambda = \rho$ and they must commute with Q.*

Proof: As $\lambda Q \rho^T = Q \iff \lambda = Q \rho Q^{-1}$ and $\lambda \in SO(3)$, then

$$(Q\rho Q^{-1})^T = (Q\rho Q^{-1})^{-1}.$$

If Q is diagonal $(Q^{-1})^2 \rho^T Q^2 = \rho^T$, and it follows that ρ^T, and hence ρ, must commute with Q. The condition $\lambda Q \rho^T = Q$ then implies $\lambda = \rho$.

□

For a subset A of $SO(3)$ denote by A^D the set of points (λ, ρ) of $SO(3) \times SO(3)$ such that $\lambda = \rho \in A$.

Corollary 4 *The isotropy subgroup of $(Q, \beta) \in GL^+(3) \times gl(3)^*$ is conjugate to a subgroup of $\mathbf{Z}_2^\tau \times SO(3)^D$.*

Thus we restrict attention to the action of $\mathbf{Z}_2 \times_s SO(3)^D$ on the space of (Q, β), with Q diagonal, given by:

$$\begin{aligned} (1, \rho) \cdot (Q, \beta) &= (\rho Q \rho^T, \rho \beta \rho^T) \\ (\tau, \rho) \cdot (Q, \beta) &= (\rho Q \rho^T, \rho Q \beta^T Q^{-1} \rho^T) \end{aligned} \tag{2.37}$$

Define also the diagonal matrix $Q^{-1/2} = (Q^{1/2})^{-1}$.

Lemma 8 *An element (σ, ρ) fixes (Q, β) under the action (2.37) if and only if:*

- *Q commutes with ρ, and*
- *$(\sigma, \rho) \in \mathbf{Z}_2 \times_s SO(3)^D$ fixes $\alpha = Q^{-1/2} \beta Q^{1/2}$ under the action of $\mathbf{Z}_2 \times_s SO(3)^D$ on $gl(3)^*$ defined by:*

$$\begin{aligned} (1, \rho) \cdot \alpha &= \rho \alpha \rho^T \\ (\tau, \rho) \cdot \alpha &= \rho \alpha^T \rho^T \end{aligned} \tag{2.38}$$

Proof : $\rho Q \rho^T = Q \Longleftrightarrow Q$ commutes with ρ.
If Q commutes with ρ:

$$\begin{aligned} \rho \beta \rho^T = \beta \quad &\Longleftrightarrow \rho Q^{-1/2} \beta Q^{1/2} \rho^T = Q^{-1/2} \beta Q^{1/2} \\ \rho Q \beta^T Q^{-1} \rho^T = \beta \quad &\Longleftrightarrow \rho (Q^{-1/2} \beta Q^{1/2})^T \rho^T = Q^{-1/2} \beta Q^{1/2} \end{aligned}$$

□

Corollary 5 *The isotropy subgroups for the action (2.37) are precisely the isotropy subgroups for the action (2.38).*

Proof: If Σ is the isotropy subgroup of (Q, β) under the action (2.37) then Σ is the intersection of the isotropy subgroup of $\alpha = Q^{-1/2} \beta Q^{1/2}$ under the action (2.38) with

$$\begin{aligned} &\mathbf{Z}_2^\tau \times SO(3) \quad &&\text{if} \quad Q = kI \\ &\mathbf{Z}_2^\tau \times O(2) \quad &&\text{if} \quad Q = \begin{pmatrix} k & & \\ & k & \\ & & n \end{pmatrix} \\ &\mathbf{Z}_2^\tau \times D_2 \quad &&\text{if} \quad Q = \begin{pmatrix} k & & \\ & l & \\ & & n \end{pmatrix} \end{aligned}$$

But all intersections of isotropy subgroups of (2.38) with any of these is another isotropy subgroup of (2.38).

Conversely, if Σ is the isotropy subgroup of α under (2.38) then it is the isotropy subgroup of $(I, Q^{-1/2}\beta Q^{1/2})$ under the action (2.37).

□

Thus the lattice of isotropy subgroups for the action of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on $GL^+(3) \times gl(3)^*$ can be identified with that of figure 2.4, with $SO(3)$ identified with $SO(3)^D$ in $SO(3) \times SO(3)$.

We end this section with a list of representatives (table 2.4) of different orbit types.

2.5 Symmetries and Riemann's Ellipsoids

It was Riemann [40] who gave for the first time a complete description of ellipsoidal figures of equilibrium for Dirichlet's problem.

As described in section 2.1 Dirichlet's problem is concerned with the motion of a homogeneous fluid mass that maintains at all times, under its own gravitation, an ellipsoidal shape. This problem can be described using two types of frames, one fixed in space and other fixed in the body and rotating with it. The frame in which the ellipsoid axes are at rest (body frame) rotates with angular velocity $\omega(t) \in \mathbf{R}^3$ with respect to the spatial frame and in the body frame the fluid has internal motions with uniform vorticity $\xi(t) \in \mathbf{R}^3$. For ellipsoidal figures of equilibrium ω and ξ are time independents.

Riemann's main result gives conditions on the angular velocity and vorticity of an ellipsoid of equilibrium. These conditions lead to the classification of ellipsoids of equilibria into S-type ellipsoids and ellipsoids of type I, II and III.

Among these ellipsoids of equilibrium one can find ellipsoids found by other authors before Riemann such as Maclaurin spheroids and Jacobi and Dedekind ellipsoids. Both Maclaurin and Jacobi ellipsoids arise from a homogeneous rotating mass with a uniform angular velocity and with no internal motion, so the vorticity equals zero. They are distinguished by the dimensions of the semi-axes lengths, a_i, of the ellipsoids. That is, while Maclaurin spheroids have two equal a_i's Jacobi ellipsoids have the three unequal a_i's. A Dedekind ellipsoid is an ellipsoid congruent to a Jacobi ellipsoid (see section 2.3).

The geometric approach we are following allows us to prove Riemann's theorem without any knowledge of the Hamiltonian. We are also able to find some of the defining relations of ellipsoids which are not of S-type.

We will describe, in this section, the symmetries of different relative equilibria ellipsoids. These symmetries are contained in $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ and lead us to the definition of two special kinds of relative equilibria, self-adjoint and symmetric-adjoint.

Isotropy subgroup	Q	β
$\mathbf{Z}_2^\tau \times SO(3)^D$	$\mathrm{diag}\,(k,k,k)$	$\mathrm{diag}\,(a,a,a)$
$\mathbf{Z}_2^\tau \times O(2)^D$	$\mathrm{diag}\,(k,k,l)$	$\mathrm{diag}\,(a,a,b)$
$\mathbf{Z}_2^\tau \times D_2^D$	$\mathrm{diag}\,(k,l,n)$	$\mathrm{diag}\,(a,b,c)$
$\widetilde{O(2)}^D$	$\mathrm{diag}\,(k,k,l)$	$\begin{pmatrix} a & -d & \\ d & a & \\ & & c \end{pmatrix}$
$\mathbf{Z}_2^\tau \times \mathbf{Z}_2^D$	$\mathrm{diag}\,(k,l,n)$	$\begin{pmatrix} a & \sqrt{\frac{k}{l}}\,d & \\ \sqrt{\frac{l}{k}}\,d & b & \\ & & c \end{pmatrix}$
$\widetilde{D_2}^D$	$\mathrm{diag}\,(k,l,n)$	$\begin{pmatrix} a & -\sqrt{\frac{k}{l}}\,d & \\ \sqrt{\frac{l}{k}}\,d & b & \\ & & c \end{pmatrix}$
$\mathbf{Z}_2^\tau$	$\mathrm{diag}\,(k,l,n)$	$\begin{pmatrix} a & \sqrt{\frac{k}{l}}\,d & \sqrt{\frac{k}{n}}\,e \\ \sqrt{\frac{l}{k}}\,d & b & \sqrt{\frac{l}{n}}\,f \\ \sqrt{\frac{n}{k}}\,e & \sqrt{\frac{n}{l}}\,f & b \end{pmatrix}$
$\mathbf{Z}_2^D$	$\mathrm{diag}\,(k,l,n)$	$\begin{pmatrix} a & d & \\ e & b & \\ & & c \end{pmatrix}$
$\widetilde{\mathbf{Z}_2}^D$	$\mathrm{diag}\,(k,l,n)$	$\begin{pmatrix} a & -\sqrt{\frac{k}{l}}\,d & -\sqrt{\frac{k}{n}}\,e \\ \sqrt{\frac{l}{k}}\,d & b & \sqrt{\frac{l}{n}}\,f \\ \sqrt{\frac{n}{k}}\,e & \sqrt{\frac{n}{l}}\,f & c \end{pmatrix}$
$\mathbf{1}$	$\mathrm{diag}\,(k,l,n)$	general matrix

Table 2.4: Isotropy subgroups for the action of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on $GL^+(3) \times gl(3)^*$ and the intersections of their fixed point spaces with $\{diag\} \times gl(3)^*$.

2.5.1 Riemann's theorem

Riemann's theorem states that an ellipsoid of equilibrium either has angular velocity, ω, and vorticity, ξ, parallel to the same principal axis of the ellipsoid or lying in the same principal plane of the ellipsoid.

The term "ellipsoid" in Riemann's theorem is general in the sense that it does not make any distinction between spherioidal and ellipsoidal shapes. Here we introduce the following terminology.

Definition 8 *A relative equilibrium is said to be:*

- *spherical if all its principal axes have the same lengths.*
- *spheroidal if two of its principal axes have the same lengths.*
- *ellipsoidal if all its principal axes have different lengths.*

Under the formalism we are using the angular velocity ω and the vorticity ξ are identified with derivatives of certain matrices of $SO(3)$ at the identity, i.e elements $\hat{\omega}$ and $\hat{\xi}$ of the Lie algebra $so(3)$ (see section 2.1). A relative equilibrium Q for the affine rigid body with angular velocity ω and vorticity ξ has the form $Q(t) = \exp(t\hat{\omega})\, Q_0 \exp(-t\hat{\lambda})$, where Q_0 is a diagonal matrix, $\hat{\omega}$, $\hat{\lambda}$, $\hat{\xi} \in so(3)$ with $\hat{\omega} \cong \omega$ and $\hat{\lambda} \cong \lambda$ and $\hat{\xi} = Q_0^{-1}\hat{\lambda}^T Q_0 - Q_0\hat{\lambda}Q_0^{-1}$ (by 2.9).

Theorem 6 (Riemann) *The angular momentum and circulation of a relative equilibrium are parallel to ω and λ respectively. If the relative equilibrium is an ellipsoid then the angular momentum and circulation are either parallel to the same principal axis of the ellipsoid or they lie in the same principal plane.*

In order to prove this theorem we will use the following result. Let a_1, a_2, a_3 be the semi-axes lengths of an ellipsoid.

Lemma 9 *If $(\omega_i\,,\,\lambda_i) \neq (0\,,\,0)$ and $(\omega_j\,,\,\lambda_j) \neq (0\,,\,0)$ $(i \neq j)$ then either $a_i = a_j$ or*

$$\alpha_1 + \alpha_2 = -\frac{1}{2}(a_i^2 + a_j^2 - 2a_k^2)$$

and (2.39)

$$\alpha_1\alpha_2 = \frac{1}{4}\left[a_k^2(a_i^2 + a_j^2) + a_i^2 a_j^2 - 3a_k^4)\right]$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ and α_1, α_2 are defined by (2.42).

Proof (theorem 6): Let $(J_1\,, J_2) = (\hat{\mu}_1\,,\,\hat{\mu}_2)$ be the momentum map value at the relative equilibrium $Q(t) = \exp(t\hat{\omega})\, Q_0 \exp(-t\hat{\lambda})$.

As the momentum map is constant along trajectories and $SO(3) \times SO(3))$ equivariant, we have:

$$\exp(-t\hat{\omega})\,\hat{\mu}_1 \exp(t\hat{\omega}) = \hat{\mu}_1$$
$$\exp(-t\hat{\lambda})\,\hat{\mu}_2 \exp(t\hat{\lambda}) = \hat{\mu}_2.$$

Differentiating these equalities we get:

$$\hat{\mu}_1 \hat{\omega} - \hat{\omega} \hat{\mu}_1 = 0$$

$$\hat{\mu}_2 \hat{\lambda} - \hat{\lambda} \hat{\mu}_2 = 0$$

which are equivalent to:

$$\begin{aligned} \mu_1 \times \omega &= 0 \\ \mu_2 \times \lambda &= 0. \end{aligned} \tag{2.40}$$

The momentum map components are the angular momentum and circulation vectors. These last equalities prove the first part of the theorem. They are equivalent to saying that the angular momentum and circulation are parallel respectively to ω and λ.

Without any loss of generality we may assume that Q_0 is diagonal: $Q_0 = D = \operatorname{diag}(a_1, a_2, a_3)$.

Differentiating at zero the expression of Q, and using the momentum map expressions (2.23) we have:

$$\begin{aligned} J_1 &= \frac{1}{2}(\dot{Q}_0 Q_0^T - Q_0 \dot{Q}_0^T) = \frac{1}{2}(\hat{\omega} D^2 + D^2 \hat{\omega} - 2 D \hat{\lambda} D) \\ &= \frac{1}{2}(\operatorname{tr}(D^2)\mathbf{I} - D^2)\omega - (\det D) D^{-1} \lambda, \end{aligned}$$

where the last equality follows by lemma 2 b) and c). For J_2 we get:

$$\begin{aligned} J_2 &= \frac{1}{2}(\dot{Q}_0^T Q_0 - {Q_0}^T \dot{Q}_0) = \frac{1}{2}(\hat{\lambda} D^2 + D^2 \hat{\lambda} - 2 D \hat{\omega} D) \\ &= \frac{1}{2}(\operatorname{tr}(D^2)\mathbf{I} - D^2)\lambda - (\det(D)) D^{-1} \omega. \end{aligned}$$

Taking

$$A = -\frac{1}{2}(\operatorname{tr}(D^2)\mathbf{I} - D^2) = \operatorname{diag}\left\{-\frac{1}{2}(a_2^2 + a_3^2), -\frac{1}{2}(a_1^2 + a_3^2), -\frac{1}{2}(a_1^2 + a_2^2)\right\}$$

$$B = (\det(D)) D^{-1} = \operatorname{diag}\{a_2 a_3, a_1 a_3, a_1 a_2\},$$

we have:

$$-\mu_1 = A\omega + B\lambda \qquad -\mu_2 = A\lambda + B\omega. \tag{2.41}$$

So, equation (2.40) is equivalent to:

$$\begin{aligned} A\omega + B\lambda &= \alpha_1 \omega \\ B\omega + A\lambda &= \alpha_2 \lambda, \end{aligned} \tag{2.42}$$

for some real numbers α_1, α_2.

The statement that an ellipsoid of equilibrium must have angular momentum and circulation either parallel to the same principal axis of the ellipsoid, or lying in the same principal plane, is equivalent to the statement that at least one of the pairs (ω_i, λ_i), where ω_i, λ_i's are the components of ω and λ, must be zero. If this is not true then by lemma 9 the relative equilibrium is either spheroidal ($a_i = a_j$) or

$$\alpha_1 + \alpha_2 = -\frac{1}{2}(a_1^2 + a_2^2 - 2a_3^2) = -\frac{1}{2}(a_2^2 + a_3^2 - 2a_1^2) = -\frac{1}{2}(a_3^2 + a_1^2 - 2a_2^2).$$

This is equivalent to $a_1 = a_2 = a_3$, that is the relative equilibrium is spherical.

□

Proof (Lemma 9): Writing the equation (2.42) in coordinates we have, for each cyclic permutation of (i,j,k) of $(1,2,3)$:

$$\begin{aligned} \left[-\frac{1}{2}(a_j^2 + a_k^2) - \alpha_1\right]\omega_i + a_j a_k \lambda_i &= 0 \\ a_j a_k \omega_i + \left[-\frac{1}{2}(a_j^2 + a_k^2) - \alpha_2\right]\lambda_i &= 0 \end{aligned} \tag{2.43}$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$.

These equations have a nonzero solution for (ω_i, λ_i) if and only if:

$$\frac{1}{4}\left[(a_j^2 + a_k^2) + 2\alpha_1\right]\left[(a_j^2 + a_k^2) + 2\alpha_2\right] - a_j^2 a_k^2 = 0$$

that is:

$$4\alpha_1\alpha_2 + 2(a_j^2 + a_k^2)(\alpha_1 + \alpha_2) + (a_j - a_k)^2(a_j + a_k)^2 = 0 \tag{2.44}$$

Taking in equation (2.44) the permutations (i,j,k) and (j,k,i) we have:

$$4\alpha_1\alpha_2 + 2(a_j^2 + a_k^2)(\alpha_1 + \alpha_2) + (a_j - a_k)^2(a_j + a_k)^2 = 0$$

$$4\alpha_1\alpha_2 + 2(a_k^2 + a_i^2)(\alpha_1 + \alpha_2) + (a_k - a_i)^2(a_k + a_i)^2 = 0.$$

Subtracting these equations we get:

$$\alpha_1 + \alpha_2 = -\frac{1}{2}(a_i^2 + a_j^2 - 2a_k^2)$$

Substituting this expression into (2.44) and solving in order to $\alpha_1\alpha_2$ we get the expression:

$$\alpha_1\alpha_2 = \frac{1}{4}\left[a_k^2(a_i^2 + a_j^2) + a_i^2 a_j^2 - 3a_k^4\right]$$

□

Riemann and subsequent authors refer to relative equilibria for which the angular velocity and vorticity are parallel to the same principal axis of the relative equilibrium as "S-type ellipsoids". The ones for which the angular momentum and vorticity lie in the same principal plane are divided into three classes, types I,II and III.

It is not hard to see that in the S-type ellipsoids case the condition on angular velocity and vorticity is equivalent to the same condition on the angular momentum and circulation.

Lemma 9 allows us to distinguish two types of ellipsoidal relative equilibria which are not of type S.

Proposition 19 *The lengths, a_1, a_2, a_3, of a relative equilibrium ellipsoid with non parallel angular momentum and circulation satisfy precisely one of the following conditions:*

1. $a_k \geq \frac{1}{2}(a_i + a_j)$.

2. $a_k \leq \frac{1}{2}|a_i - a_j|$.

The indices i, j are the ones for which $(\omega_i, \lambda_i) \neq 0$ and $(\omega_j, \lambda_j) \neq 0$, for $i \neq j$.

Proof: The ellipsoidal relative equilibrium of the hypothesis of this proposition have, for some i, j, $(i \neq j)$, $(\omega_i, \lambda_i) \neq 0$ and $(\omega_j, \lambda_j) \neq 0$. So it must verify equations (2.39) of lemma 9 for α_1, α_2 real. Thus

$$(\alpha_1 - \alpha_2)^2 \geq 0 \Longleftrightarrow (\alpha_1 + \alpha_2)^2 \geq 4\,\alpha_1\,\alpha_2.$$

Substituting the expressions for $(\alpha_1 + \alpha_2)$ and $(\alpha_1\,\alpha_2)$ given by (2.39) into the last expression we have:

$$\left[a_k^2 - \frac{1}{2}(a_i^2 + a_j^2)\right]^2 \geq \left[a_i^2\,a_j^2 + a_j^2\,a_k^2 + a_k^2\,a_i^2 - 3\,a_k^4\right]$$

Rearranging and factoring gives:

$$\left[4\,a_k^2 - (a_i - a_j)^2\right]\left[4\,a_k^2 - (a_i + a_j)^2\right] \geq 0.$$

The study of the sign of this expression gives the result.

□

Riemann's ellipsoids of type I are defined by condition 1. of the last proposition while types II and III are defined by condition 2. (see Chandrasekhar [10], Chapter 5, §51).

2.5.2 Symmetries of relative equilibria

Here we introduce the definitions of self-adjoint and symmetric-adjoint relative equilibria. These equilibria have symmetry groups contained in $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$. Some results on the angular momentum and circulation of these equilibria are established as well as for the semi-axes lengths of ellipsoids of equilibria of these kinds.

We finish this section with the description of the symmetries of all possible relative equilibria of the affine rigid body.

Definition 9 *We say that a motion $Q(t)$ is self-adjoint if*

$$[Q(t)]^T = Q(t) \qquad \text{for all} \quad t \tag{2.45}$$

and that $Q(t)$ is symmetric-adjoint if

$$[Q(t)]^T = \rho_1\,Q(t)\,\rho_2^T \qquad \text{for all} \quad t \tag{2.46}$$

for a pair of elements ρ_i of $SO(3)$ satisfying $\rho_i^2 = 1$ and $\rho_i \neq 1$.

Note that a motion is self-adjoint if and only if its symmetry group contains $\mathbf{Z}_2^\tau$ and is symmetric-adjoint if and only if the symmetry group contains a group conjugate to $\widetilde{\mathbf{Z}}_2$. Note also that the definition of self-adjoint given here differs from the one used by Chandrasekhar [10]. This author defines the equilibrium ellipsoid $Q(t) = \exp(t\hat{\omega})\, Q_0 \exp(-t\hat{\lambda})$ to be self-adjoint if and only if $\hat{\omega} = \hat{\lambda}$.

The next two results characterize self-adjoint and symmetric-adjoint relative equilibria. The first one is a version of Riemann's theorem for self-adjoint and symmetric adjoint relative equilibria. The second result is on the lengths of the principal axes of self-adjoint and symmetric adjoint relative equilibria ellipsoids.

Lemma 10 *Let $Q(t) = \exp(t\hat{\omega})\, Q_0 \exp(-t\hat{\lambda})$ be a relative equilibrium with momentum map value $(J_1, J_2) = (\hat{\mu}_1, \hat{\mu}_2)$.*

i) If $Q(t)$ is self-adjoint then $\omega = \lambda$ and $\mu_1 = \mu_2$.

ii) If $Q(t)$ is symmetric-adjoint then $(\lambda + \omega)$ and $(\lambda - \omega)$ are orthogonal to each other.

If $Q(t)$ is a symmetric-adjoint ellipsoid then $(\lambda + \omega)$ and $(\lambda - \omega)$ are parallel to different axes.

If $Q(t)$ is a symmetric-adjoint spheroid then either $(\lambda + \omega)$ and $(\lambda - \omega)$ both lie in the plane orthogonal to the axis of rotation, or one of them lies in that plane and the other is parallel to the axis of rotation.

Proof: Assume Q_0 is diagonal. Adding and subtracting equations (2.41) we get:

$$-(\mu_1 + \mu_2) = (A + B)(\omega + \lambda) \tag{2.47}$$
$$-(\mu_1 - \mu_2) = (A - B)(\omega - \lambda) \tag{2.48}$$

where

$$A + B = \operatorname{diag}\left(-\frac{1}{2}(a_2 - a_3)^2, -\frac{1}{2}(a_3 - a_1)^2, -\frac{1}{2}(a_1 - a_2)^2\right)$$

$$A - B = \operatorname{diag}\left(-\frac{1}{2}(a_2 + a_3)^2, -\frac{1}{2}(a_3 + a_1)^2, -\frac{1}{2}(a_1 + a_2)^2\right)$$

i) If $Q(t)$ is self-adjoint then, by equivariance of the momentum map, $\tau \cdot (J_1, J_2) = (J_2, J_1)$ ($\tau \neq 1$ and $\tau \in \mathbf{Z}_2$) and so $\mu_1 = \mu_2$. Equation (2.48) then implies $\omega = \lambda$.

ii) Recall from section 2.4.2 (the derivation of the isotropy lattice) that when Q is diagonal and $\rho_1\, Q\, \rho_2^T = Q$ then $\rho_1 = \rho_2 = \rho$, say, and ρ commutes with Q.

If $\rho^2 = \mathrm{I}$ this implies that ρ is a rotation by π about one of the principal axes of Q, if Q is ellipsoidal, or about the axis of symmetry or an axis perpendicular to that, if Q is spheroidal.

If $Q(t)$ is symmetric-adjoint then equivariance of the momentum map implies that $\rho\, \mu_1 = \mu_2$ and so

$$\rho(\mu_1 + \mu_2) = (\mu_1 + \mu_2)$$
$$\rho(\mu_1 - \mu_2) = -(\mu_1 - \mu_2)$$

Moreover, ρ commutes with A and B and so equations (2.47) and (2.48) give:

$$(A+B)\,\rho\,(\omega+\lambda) = (A+B)\,(\omega+\lambda) \tag{2.49}$$

$$(A-B)\,\rho\,(\omega-\lambda) = -(A-B)\,(\omega-\lambda) \tag{2.50}$$

Since $(A-B)$ is invertible (2.50) implies $\rho\,(\omega-\lambda) = -(\omega-\lambda)$. If Q is ellipsoidal then $(A+B)$ is also invertible and so $\rho\,(\omega+\lambda) = (\omega+\lambda)$. Thus $(\omega+\lambda)$ must be directed along the axis of rotation of ρ, a principal axis of Q, and $\omega-\lambda$ perpendicular to it. From Riemann's theorem ω and λ must lie in the same principal plane. So $\omega-\lambda$ must also be parallel to a principal axis.

For the symmetric adjoint spheroid case we have $\rho\,(\omega-\lambda) = -(\omega-\lambda)$ and $(\omega+\lambda)$ orthogonal to $(\omega-\lambda)$ and hence the result.

□

Remarks: A self-adjoint relative equilibrium may have λ and ω aligned with the same principal axis, or they may lie in a principal plane.

A symmetric-adjoint relative equilibrium does not have λ and ω parallel unless either $\lambda=\omega$ or $\lambda=-\omega$. In these cases λ and ω must be parallel to the same principal axis (or orthogonal to the axis of symmetry, in the case of a spheroid). In the first case the relative equilibrium is also self-adjoint.

Proposition 20 *Let a_1, a_2 and a_3 be the lengths of the principal axes of an ellipsoidal relative equilibrium for which $(\omega_i,\lambda_i) \neq (0,0)$ and $(\omega_j,\lambda_j) \neq (0,0)$, where $i \neq j$.*

1. *If the relative equilibrium is self-adjoint then:*

$$a_k = \frac{1}{2}\,(a_i + a_j).$$

2. *If the relative equilibrium is symmetric-adjoint then:*

$$a_k = \frac{1}{2}\,|a_i - a_j|.$$

Proof: 1) By the last lemma if the relative equilibrium is self-adjoint then $\lambda=\omega$. If $(\omega_i,\lambda_i) \neq (0,0)$ then equation (2.43) implies:

$$\alpha_1 = \alpha_2 = -\frac{1}{2}(a_j - a_k)^2.$$

Similarly, if $(\omega_j,\lambda_j) \neq (0,0)$ then

$$\alpha_1 = \alpha_2 = -\frac{1}{2}(a_i - a_k)^2.$$

So $-\frac{1}{2}(a_j - a_k)^2 = -\frac{1}{2}(a_i - a_k)^2$ and hence

$$a_k = \frac{1}{2}(a_i + a_j).$$

2) If the relative equilibrium is symmetric-adjoint then there exists ρ such that $\mu_2 = \rho\,\mu_1$ and $\lambda = \rho\,\omega$. It follows from $\mu_1 = \alpha_1\,\omega$, $\mu_2 = \alpha_2\,\lambda$ that $\alpha_1 = \alpha_2$. From equations (2.47), (2.48) we have:

$$\alpha\,(\omega + \lambda) = (A + B)\,(\omega + \lambda)$$

$$\alpha\,(\omega - \lambda) = (A - B)\,(\omega - \lambda).$$

If $(\omega_i, \lambda_i) \neq (0,0)$ and $(\omega_j, \lambda_j) \neq (0,0)$ then either

$$\alpha = -\frac{1}{2}(a_j - a_k)^2 = -\frac{1}{2}(a_i + a_k)^2$$

or

$$\alpha = -\frac{1}{2}(a_i - a_k)^2 = -\frac{1}{2}(a_j + a_k)^2.$$

The first case leads to $a_k = \dfrac{-1}{2}(a_i - a_j)$ and the second to $a_k = \dfrac{1}{2}(a_i - a_j)$. □

Remark: From the definition of ellipsoids which are not S-type (proposition 19) we conclude that the self-adjoint relative equilibria which are not of S-type occur on the boundary (in (a_1, a_2, a_3)-space) of the Riemann's ellipsoids of type I and that the symmetric-adjoint relative equilibria which are not of S-type occur on the boundary of the Riemann ellipsoids of types II and III.

Based on Riemann's theorem and consequences discussed above we now describe relative equilibria that can occur with symmetry groups conjugate to those listed in table 2.4.

Lemma 11 *For a relative equilibrium:*

$$Q(t) = \exp\,(t\Omega)\,Q_0\,\exp\,(-t\Lambda)$$

the following hold:

i) $P_0 = \dot{Q}(0) = \Omega Q_0 - Q_0 \Lambda$.

ii) In body coordinates:

$$i_L(Q_0, P_0) = (Q_0, Q_0^T \Omega Q_0 - Q_0^T Q_0 \Lambda).$$

iii) The momentum map, $J = (J_1, J_2)$, *in body coordinates is:*

$$J_1(Q_0, P_0) = \frac{1}{2}\left[\Omega Q_0 Q_0^T + Q_0 Q_0^T \Omega\right] - Q_0 \Lambda Q_0^T$$

$$J_2(Q_0, P_0) = \frac{1}{2}\left[\Lambda Q_0^T Q_0 + Q_0^T Q_0 \Lambda\right] - Q_0^T \Omega Q_0.$$

Proof:
i) Differentiating the expression of Q at $t = 0$ we get the result.

ii) Body coordinates are coordinates on $GL^+(3) \times gl(3)^*$ corresponding to the left identification of $T^*GL^+(3)$ with $GL^+(3) \times gl(3)^*$. Thus from the expression of i_L obtained in subsection 2.2.2, page 59, we have:

$$i_L(Q_0, P_0) = (Q_0, Q_0^T P_0) = (Q_0, Q_0^T \Omega Q_0 - Q_0^T Q_0 \Lambda).$$

iii) From the expression of the momentum map in body coordinates given in proposition 12 , equation (2.24), with $\beta = Q_0^T \Omega Q_0 - Q_0^T Q_0 \Lambda$, we get:

$$J_1(Q_0, \beta) = \frac{1}{2}(Q_0^{-T} \beta Q_0^T - Q_0 \beta^T Q_0^{-1}) = \frac{1}{2}\left[\Omega Q_0 Q_0^T + Q_0 Q_0^T \Omega\right] - Q_0 \Lambda Q_0^T$$

$$J_2(Q_0, \beta) = \frac{1}{2}(\beta^T - \beta) = \frac{1}{2}\left[\Lambda Q_0^T Q_0 + Q_0^T Q_0 \Lambda\right] - Q_0^T \Omega Q_0$$

□

When considering relative equilibria we may always assume that Q_0 is diagonal, $Q_0 = \operatorname{diag}(k, l, n)$. With this assumption if the relative equilibrium has symmetry group Σ then $i_L(Q_0, P_0) = (Q_0, Q_0^T \Omega Q_0 - Q_0^T Q_0 \Lambda) = (Q_0, \beta)$ must lie in the fixed point set given in table 2.4.

When $Q_0 = \operatorname{diag}(k, l, n)$ and $\hat{\omega} = \Omega$, $\hat{\lambda} = \Lambda$, where $\omega = (\omega_1, \omega_2, \omega_3)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ we have for $\beta = Q_0 \Omega Q_0 - Q_0^2 \Lambda$:

$$\beta = \begin{bmatrix} 0 & -kl\omega_3 & kn\omega_2 \\ kl\omega_3 & 0 & -ln\omega_1 \\ -kn\omega_2 & ln\omega_1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -k^2\lambda_3 & k^2\lambda_2 \\ l^2\lambda_3 & 0 & -l^2\lambda_1 \\ -n^2\lambda_2 & n^2\lambda_1 & 0 \end{bmatrix} \tag{2.51}$$

Consider now the different groups in table 2.4:

1. $\mathbf{Z}_2^\tau \times SO(3)^D$, $\quad \mathbf{Z}_2^\tau \times O(2)^D$, $\quad \mathbf{Z}_2^\tau \times D_2^D$

   In each of these cases comparison of β with the corresponding entry in column 3 of table 2.4 leads to the conclusion that $\dot{Q} = 0$. So the only possible relative equilibria with these symmetry groups are, in fact, equilibria. They correspond respectively to spherical, spheroidal and ellipsoidal equilibria.

2. $\widetilde{O(2)}^D$

   From table 2.4 we must have $\beta = \begin{bmatrix} a & -d & 0 \\ d & a & 0 \\ 0 & 0 & b \end{bmatrix}$. Thus solving the equation resulting from making equal β given by this form and given by (2.51) we get:

$$\dot{Q}_0 = \begin{bmatrix} 0 & -k(\omega_3 - \lambda_3) & 0 \\ k(\omega_3 - \lambda_3) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   since $\beta = Q_0 \dot{Q}_0$. Note that this motion depends only on the differece $\Omega - \Lambda$, so two trajectories with the same $\Omega - \Lambda$ are equal.

Using part iii) of lemma 11 we have, for $J_1 = \hat{\mu}_1$ and $J_2 = \hat{\mu}_2$:

$$\mu_1 = (0\,,\,0\,,\,k^2(\omega_3 - \lambda_3))$$

$$\mu_2 = (0\,,\,0\,,\,-k^2(\omega_3 - \lambda_3))$$

and so $J_1 = -J_2$.

These solutions are spheroids rotating about their axes of symmetry and can be identified with Maclaurin spheroids of self-gravitating fluids. Conversely, every spheroidal relative equilibrium rotating about its axis of symmetry has symmetry group conjugate to $\widetilde{O(2)}^D$.

3. $\mathbf{Z}_2^\tau \times \mathbf{Z}_2^D$

   The $\mathbf{Z}_2^\tau$ component of the symmetry group means that the corresponding relative equilibria are self-adjoint and so, by lemma 11, $\lambda = \omega$ and $\mu_1 = \mu_2$. The form of β in column 3 of table 2.4 shows that $\omega_2 = \omega_1 = \lambda_2 = \lambda_1 = 0$, that is λ and ω must be parallel to a principal axis of the ellisoidal relative equilibrium. So the relative equilibria with this symmetry are self-adjoint S-type ellipsoids. Conversely every self-adjoint S-type ellipsoid has symmetry group conjugate to $\mathbf{Z}_2^\tau \times \mathbf{Z}_2^D$.

4. $\widetilde{D_2}^D$

   This group contains $\widetilde{\mathbf{Z}_2}^D$ and so the corresponding relative equilibria are symmetric adjoint. The form of β in table 2.4 indicates that they must also be of S-type. Conversely every symmetric adjoint relative equilibrium of S-type has symmetry group conjugate to $\widetilde{D_2}^D$.

5. $\mathbf{Z}_2^D$ These are S-type ellipsoids which do not have higher symmetries.

6. $\mathbf{Z}_2^\tau$ These are self-adjoint relative equilibria which are not of S-type. By the remark following proposition 20 these ellipsoids are therefore of type I.

7. $\widetilde{\mathbf{Z}_2}^D$

   These are the symmetric adjoint relative equilibria of types II and III.

8. **1**

   General relative equilibria of types I, II and III.

Table 2.5 summarizes the results just obtained.

Isotropy Subgroup	Q	Description of relative equilibria
$\mathbf{Z}_2^\tau \times SO(3)^D$	$\mathrm{diag}\,(k,k,k)$	Spherical equilibrium
$\mathbf{Z}_2^\tau \times O(2)^D$	$\mathrm{diag}\,(k,k,n)$	Spheroidal equilibrium
$\mathbf{Z}_2^\tau \times D_2^D$	$\mathrm{diag}\,(k,l,n)$	Ellipsoidal equilibrium
$\widetilde{O(2)}^D$	$\mathrm{diag}\,(k,k,n)$	Maclaurin spheroid
$\mathbf{Z}_2^\tau \times \mathbf{Z}_2^D$	$\mathrm{diag}\,(k,l,n)$	Self-adjoint, S-type ellipsoid
$\widetilde{D_2}^D$	$\mathrm{diag}\,(k,l,n)$	Symmetric-adjoint, S-type ellipsoid
$\mathbf{Z}_2^D$	$\mathrm{diag}\,(k,l,n)$	General S-type ellipsoid
$\mathbf{Z}_2^\tau$	$\mathrm{diag}\,(k,l,n)$	Self-adjoint, type I ellipsoid
$\widetilde{\mathbf{Z}_2}^D$	$\mathrm{diag}\,(k,l,n)$	Symmetric-adjoint, type II or III ellipsoids
$\mathbf{1}$	$\mathrm{diag}\,(k,l,n)$	General ellipsoids of types I, II or III

Table 2.5: Symmetries of Riemann's ellipsoids

2.6 Isotropy Subgroups Representations on Symplectic Slices

In this section we determine the representation of the isotropy subgroup of a point p of the phase space, $\mathcal{P}$, on the tangent space of $\mathcal{P}$ at the point, as well as its representation on the symplectic slice. The representation on the symplectic slice will be determined for all isotropy subgroups of the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ action on $GL^+(3) \times gl(3)^*$ listed in table 2.4.

In order to find the representation of the isotropy subgroup of a point p of the phase space on the symplectic slice Y we will use the character theory of representations and the decomposition of the tangent space at the point p, due to Montaldi, Roberts and Stewart [38] referred to in the first chapter.

Let us recall the main ingredients. Let Γ be a group acting symplectically on $\mathcal{P}$. The tangent space $T_p\mathcal{P}$ decomposes into $T_p\mathcal{P} \cong W \oplus X \oplus Y \oplus Z$ where

$$\begin{aligned} W &= Ker\ dJ_p \cap T_p(G \cdot p) \\ X &= T_p(G \cdot p)/W \\ Y &= Ker\ dJ_p/W \\ Z &= T_pP/Ker\ dJ_p + T_p(G \cdot p) \end{aligned}$$

Let γ_μ be the Lie algebra of the isotropy subgroup Σ_μ, where $\mu = J(p)$, and γ_p and γ respectively the Lie algebra of the isotropy subgroup of p and the Lie algebra of Γ. From proposition 4.3 of Montaldi, Roberts and Stewart [38] we have

1) W and γ_μ/γ_p are isomorphic representations of the isotropy subgroup Σ_p of p and $W \oplus Z$ is symplectically isomorphic to $\gamma_\mu/\gamma_p \otimes \mathbf{C}$.

2) X and γ/γ_p are isomorphic symplectic representations of Σ_p.

Associated to each symplectic representation there is a unitary representation. The character of a unitary representation ρ is a function $\chi : \Gamma \to \mathbf{C}$ given by $\chi(\gamma) = \mathrm{tr}\,\rho(\gamma)$ where $\rho_\gamma(x) = \gamma x$. The function χ is constant on conjugacy classes, since $\chi(\delta^{-1}\gamma\delta) = \mathrm{tr}\,(\rho_\delta^{-1}\rho_\gamma\rho_\delta) = \chi(\gamma)$.

Denoting by $\chi(Y)$ the character of the representation of Σ_p on Y we have

$$\chi(Y) = \chi(T_p\mathcal{P}) - \chi(\gamma_\mu/\gamma_p \otimes \mathbf{C}) - \chi(\gamma/\gamma_\mu) \tag{2.52}$$

The characters of the right hand side of this equation are in principle computable, giving the representation of Σ_p on Y.

2.6.1 Representation of $\Sigma_{(Q,\beta)}$ on $T_{(Q,\beta)}\mathcal{P}$

In order to show that the representation of an isotropy subgroup of the point (Q, β) on the tangent space at the point, $T_{(Q,\beta)}\mathcal{P}$, is isomorphic to its representation on $gl(3) \times gl(3)^*$ given in proposition 21, we will consider a suitable action of the semidirect product $\mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$ on $GL^+(3)$. This action induces an action on

$T_{(Q,\beta)}(T^*GL^+(3))$ which is given by lemma 12 where $T_{(Q,\beta)}(T^*GL^+(3))$ has been identified with $gl(3) \times gl(3)^*$. Denoting by $\Sigma_{(Q,\beta)}$ and $\tilde{\Sigma}_{(Q,\beta)}$ the isotropy subgroups of (Q,β) respectively for the action of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ and $\mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$ it is then shown that the representation of $\Sigma_{(Q,\beta)}$ is isomorphic to the representation of $\tilde{\Sigma}_{(Q,\beta)}$ on $gl(3) \times gl(3)^*$.

Proposition 21 *The representation of $\Sigma_{(Q,\beta)}$ on $T_{(Q,\beta)}\mathcal{P}$ is isomorphic to its representation on $gl(3) \times gl(3)^*$ defined by*

$$(\sigma, \lambda, \rho) \cdot (u, v) = \begin{cases} (\lambda u \rho^T, \rho v \rho^T) & \text{for} \quad \sigma = 1 \\ (\lambda u^T \rho^T, \rho v^T \rho^T) & \text{for} \quad \sigma = \tau \end{cases} \tag{2.53}$$

where $(\sigma, \lambda, \rho) \in \Sigma_{(Q,\beta)} \subset \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$.

Let us define the following actions:

1. The $GL^+(3) \times GL^+(3)$-action on $GL^+(3)$, Φ', is defined as

   $$\Phi'_{(\lambda,\rho)}(Q) = \lambda Q \rho^{-1}$$

   where $(\lambda, \rho) \in GL^+(3) \times GL^+(3)$ and $Q \in GL^+(3)$.

2. The $\mathbf{Z}_2$-action on $GL^+(3) \times GL^+(3)$ is defined by

   $$\tau \cdot (\lambda, \rho) = (\rho^{-T}, \lambda^{-T})$$

   where $\mathbf{Z}_2 = \{1, \tau\}$.

3. The $\mathbf{Z}_2$-action on $GL^+(3)$ is Θ_σ, defined on page 61.

We can define the semi-direct product $\mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$ where the multiplication rule is given by (2.28), where there now $\sigma_1, \sigma_2 \in \mathbf{Z}_2$ and $(\lambda_i, \rho_i) \in GL^+(3) \times GL^+(3)$ $(i = 1, 2)$. The next proposition gives the action of this semi-direct product on $T^*GL^+(3)$.

Proposition 22 *The $\mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$-action on $T^*GL^+(3)$ identified with $GL^+(3) \times gl(3)^*$, using the left action on $GL^+(3)$, is*

$$\phi'_{(\sigma,(\lambda,\rho))}(Q, \beta) = \begin{cases} (\lambda Q \rho^{-1}, \rho^{-T} \beta \rho^T) & \text{for} \quad \sigma = 1 \\ (\lambda Q \rho^{-1}, \rho^{-T} Q \beta^T Q^{-1} \rho^T) & \text{for} \quad \sigma = \tau \end{cases}$$

Proof:

The action of $(1, \lambda, \rho)$ on $\mathcal{P} = GL^+(3) \times gl(3)^*$ is the lifted action, say Φ'^*, of Φ' to $GL^+(3)$. Following exactly the proof of proposition 13 we have:

$$\begin{aligned} \Phi'^*_{(\lambda,\rho)}(Q, \beta) &= i_L \circ (T\Phi'_{(\lambda^{-1},\rho^{-1})})^* \circ i_L^{-1}(Q, \beta) \\ &= \left(\lambda Q \rho^{-1}, \ll (T\Phi'_{(\lambda^{-1},\rho^{-1})})^*(Q^{-T}\beta), T_e L_{\lambda Q \rho^{-1}}(\xi) \gg\right) \end{aligned}$$

where i_L denotes the left identification of $T^*GL^+(3)$ with $gl(3)\times gl(3)^*$ given by (2.21). Also

$$
\begin{aligned}
&\ll (T\Phi'_{(\lambda^{-1},\rho^{-1})})^*(Q^{-T}\beta), T_eL_{\lambda Q\rho^{-1}}(\xi) \gg \\
&\qquad = \ll Q^{-T}\beta, \left(T_{\lambda Q\rho^{-1}}\Phi'_{(\lambda^{-1},\rho^{-1})}\circ T_eL_{\lambda Q\rho^{-1}}\right)(\xi) \gg \\
&\qquad = \ll Q^{-T}\beta, T_{\lambda Q\rho^{-1}}\Phi'_{(\lambda^{-1},\rho^{-1})}(\lambda Q\rho^{-1}\xi) \gg \\
&\qquad = \ll Q^{-T}\beta, Q\rho^{-1}\xi\rho \gg = \mathrm{tr}\,(\beta^T Q^{-1}Q\rho^{-1}\xi\rho) \\
&\qquad = \mathrm{tr}\,(\rho\beta^T\rho^{-1}\xi) = \mathrm{tr}\,(\beta^T Ad_{\rho^{-1}}\xi) = \ll Ad^*_{\rho^T}\beta, \xi \gg .
\end{aligned}
$$

So $\Phi'^*_{(\lambda,\rho)}(Q,\beta) = (\lambda Q\rho^{-1}, \rho^{-T}\beta\rho^T)$.

The lift of the $\mathbf{Z}_2$ action on $GL^+(3)$, Θ^*_σ, calculated on page 61, proposition 13, is the identity for $\sigma = 1$ and for $\sigma = \tau$

$$\Theta^*_\tau(Q,\beta) = (Q^T, Q\beta^T Q^{-1}) = (Q^T, Ad^*_{Q^T}\beta^T).$$

Let us prove that the following relation between $\Phi'^*_{(\lambda,\rho)}$ and Θ^*_σ

$$\left(\Theta^*_\tau \circ \Phi'^*_{(\lambda,\rho)} \circ \Theta^*_{\tau^{-1}}\right)(Q,\beta) = \Phi'^*_{(\tau\cdot(\lambda,\rho))}(Q,\beta),$$

where $\tau\cdot(\lambda,\rho) = (\rho^{-T},\lambda^{-T})$.

This relation is immediate for $\sigma = 1$ and for $\sigma = \tau$ we have

$$\Phi'^*_{(\tau\cdot(\lambda,\rho))}(Q,\beta) = \Phi'^*_{(\rho^{-T},\lambda^{-T})}(Q,\beta) = (\rho^{-T}Q\lambda^T, \lambda\beta\lambda^{-1}).$$

So

$$
\begin{aligned}
\left(\Theta^*_\tau \circ \Phi'^*_{(\lambda,\rho)} \circ \Theta^*_{\tau^{-1}}\right)(Q,\beta) &= \Theta^*_\tau \circ \Phi'^*_{(\lambda,\rho)}(Q^T, Q\beta^T Q^{-1}) \\
&= \Theta^*_\tau(\lambda Q^T\rho^{-1}, \rho^{-T}Q\beta^T Q^{-1}\rho^T) \\
&= (\rho^{-T}Q\lambda^T, \lambda Q^T\rho^{-1}\rho Q^{-T}\beta Q^T\rho^{-1}\rho Q^{-T}\lambda^{-1}) \\
&= \Phi'^*_{(\tau\cdot(\lambda,\rho))}(Q,\beta).
\end{aligned}
$$

The multiplication rule in $\mathbf{Z}_2\times_s(GL^+(3)\times GL^+(3))$ is the same as the one applied in the proof of theorem 5. So, by the proof of this theorem, the desired action is given by

$$\phi'_{(\sigma,(\lambda,\rho))}(Q,\beta) = \Phi'^*_{(\lambda,\rho)}\left(\Theta^*_\sigma(Q,\beta)\right) = \begin{cases} (\lambda Q\rho^{-1}, \rho^{-T}\beta\rho^T) & \text{for}\quad \sigma = 1 \\ (\lambda Q\rho^{-1}, \rho^{-T}Q\beta^T Q^{-1}\rho^T) & \text{for}\quad \sigma = \tau. \end{cases}$$

□

The tangent space $T_{(Q,\beta)}\mathcal{P}$ is identified with $gl(3) \times gl(3)^*$ by identifying $(u,v) \in gl(3) \times gl(3)^*$ with the tangent to the curve $(Q\exp(tu), \beta + tv)$ of $\mathcal{P}$ at (Q,β). That is

$$
\begin{aligned}
gl(3) \times gl(3)^* &\rightarrow T_{(Q,\beta)}\mathcal{P} \\
(u,v) &\mapsto (Qu, v).
\end{aligned}
\tag{2.54}
$$

Lemma 12 *Let $\bar{\Sigma}_{(Q,\beta)}$ denote the isotropy subgroup of the $\mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$-action on $T_{(Q,\beta)}\mathcal{P}$, identified with $gl(3) \times gl(3)^*$. The action of $\bar{\Sigma}_{(Q,\beta)}$ on $gl(3) \times gl(3)^*$ is given by:*

$$
\Psi'_{(\sigma,(\lambda,\rho))}(u,v) = \begin{cases} (Q^{-1}\lambda Q u\rho^{-1}\,,\ \rho^{-T} v \rho^T) & \text{for } \sigma = 1 \\ (Q^{-1}\lambda u^T Q^T \rho^{-1}\,,\ \rho^{-T} Q \left\{v^T + [u, \beta^T]\right\} Q^{-1}\rho^T) & \text{for } \sigma = \tau. \end{cases}
\tag{2.55}
$$

Corollary 6 *The action of $\Sigma_{(Q,\beta)} = \bar{\Sigma}_{(Q,\beta)} \cap (\mathbf{Z}_2 \times_s (SO(3) \times SO(3)))$ on $gl(3) \times gl(3)^*$ is given by*

$$
\Psi_{(\sigma,(\lambda,\rho))}(u,v) = \begin{cases} (Q^{-1}\lambda Q u\rho^T\,,\ \rho v \rho^T) & \text{for } \sigma = 1 \\ (Q^{-1}\lambda u^T Q^T \rho^T\,,\ \rho Q \left\{v^T + [u, \beta^T]\right\} Q^{-1}\rho^T) & \text{for } \sigma = \tau. \end{cases}
\tag{2.56}
$$

Proof (lemma 12):

$$
\begin{aligned}
\Psi'_{(\sigma,(\lambda,\rho))}(u,v) &= \frac{d}{dt}\phi'_{(\sigma,(\lambda,\rho))}(Q\exp(tu), \beta + tv)\Big|_{t=0} \\
&= \begin{cases} (\lambda Q u\rho^{-1}\,,\ \rho^{-T} v \rho^T) & \text{for} \quad \sigma = 1 \\ (\lambda u^T Q^T \rho^{-1}\,,\ \rho^{-T} Q \left\{v^T + [u, \beta^T]\right\} Q^{-1}\rho^T) & \text{for} \quad \sigma = \tau \end{cases} \\
&= \begin{cases} (QQ^{-1}\lambda Q u\rho^{-1}\,,\ \rho^{-T} v \rho^T) & \text{for} \quad \sigma = 1 \\ (QQ^{-1}\lambda u^T Q^T \rho^{-1}\,,\ \rho^{-T} Q \left\{v^T + [u, \beta^T]\right\} Q^{-1}\rho^T) & \text{for} \quad \sigma = \tau. \end{cases}
\end{aligned}
$$

Applying the identification (2.54) we get the result.

□

Let us define the following action of the whole group $\mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$ on $gl(3) \times gl(3)^*$

$$
(\sigma, \lambda, \rho)_{(1,\beta)} \cdot (u,v) = \begin{cases} (\lambda u \rho^{-1}\,,\ \rho^{-T} v \rho^T) & \sigma = 1 \\ (\lambda u^T \rho^{-1}\,,\ \rho^{-T}\left\{v^T + [u, \beta^T]\right\}\rho^T) & \sigma = \tau \end{cases}
\tag{2.57}
$$

where $(\sigma, \lambda, \rho) \in \mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$.

Denote now the action of $\tilde{\Sigma}_{(Q,\beta)}$ on $gl(3) \times gl(3)^*$, $\Psi'_{(\sigma,(\lambda,\rho))}$, by the same kind of notation as the one for $\mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$-action but using the subscript (Q, β) instead, i.e

$$(\sigma, \lambda, \rho)_{(Q,\beta)} \cdot (u, v) = \begin{cases} (Q^{-1}\lambda Q u \rho^{-1}\,,\ \rho^{-T} v \rho^T) & \sigma = 1 \\ (Q^{-1}\lambda u^T Q^T \rho^{-1}\,,\ \rho^{-T} Q \left\{ v^T + [u, \beta^T] \right\} Q^{-1} \rho^T) & \sigma = \tau. \end{cases} \tag{2.58}$$

Note that the formulae (2.57) and (2.58) are consistent in the sense that the action of $\tilde{\Sigma}_{(1,\beta)}$ on $T_{(1,\beta)}\mathcal{P}$ given by (2.58) is equal to the restriction of the $\mathbf{Z}_2 \times_s (GL^+(3) \times GL^+(3))$ action given by (2.57).

The property in the next lemma is the key one needed to prove the isomorphism between the $\Sigma_{(Q,\beta)}$-action given in corollary 6 and the one of proposition 21.

Lemma 13 *For $\sigma = 1, \tau \in \mathbf{Z}_2$:*

$$(\sigma, \lambda, \rho)_{(Q,\beta)}\,(u, v) = (1, Q^{-1}, 1)_{(1,\beta)} \cdot (\sigma, \lambda, \rho)_{(1,\beta)} \cdot (1, Q, 1)_{(1,\beta)}\,(u, v) \tag{2.59}$$

Proof : From the definition of the action and the multiplication rule (2.28) we have that the right hand side of (2.59) is just

$$(1,\, Q^{-1}\lambda Q,\, \rho)_{(1,\beta)} \cdot (u, v) \qquad \text{for} \quad \sigma = 1 \tag{2.60}$$

$$(\tau,\, Q^{-1}\lambda,\, \rho Q^{-T})_{(1,\beta)} \cdot (u, v) \qquad \text{for} \quad \sigma = \tau. \tag{2.61}$$

So

- For $\sigma = 1$ we have

$$\begin{aligned} (1, Q^{-1}\lambda Q, \rho)_{(1,\beta)} \cdot (u, v) &= (Q^{-1}\lambda Q u \rho^{-1}, \rho^{-T} v \rho^T) && \text{from (2.57)} \\ &= (1, \lambda, \rho)_{(Q,\beta)} \cdot (u, v) && \text{from (2.58).} \end{aligned}$$

- For $\sigma = \tau$ we have

$$\begin{aligned} (\tau, Q^{-1}\lambda, \rho Q^{-T})_{(1,\beta)} \cdot (u, v) &= (Q^{-1}\lambda u^T Q^T \rho^{-1}, \rho^{-T} Q \{v^T + [u, \beta^T]\} Q^{-1} \rho^T) \\ &= (\tau, \lambda, \rho)_{(Q,\beta)} \cdot (u, v) \end{aligned}$$

  where the first equality follows from (2.57) and the second from (2.58).

□

Corollary 7 *The representation of $\Sigma_{(Q,\beta)}$ on $T_{(Q,\beta)}\mathcal{P}$ is isomorphic to its representation on $gl(3) \times gl(3)^*$ defined by*

$$(\sigma, \lambda, \rho) \cdot (u, v) = \begin{cases} (\lambda u \rho^T, \rho v \rho^T) & \textit{for} \quad \sigma = 1 \\ (\lambda u^T \rho^T, \rho\{v^T + [u, \beta^T]\}\rho^T) & \textit{for} \quad \sigma = \tau \end{cases} \tag{2.62}$$

Proof: Restricting the action defined by (2.57) to $\Sigma_{(Q,\beta)}$ the isomorphism is given by (2.59) for $(1, Q, 1)$.

□

Proof (proposition 21):

It is sufficient to show that the representation of $\Sigma_{(Q,\beta)}$ defined in corollary 7 by (2.62) is equivalent to that defined by (2.53).

With respect to a suitable basis for $gl(3) \times gl(3)^*$ the matrix of (σ, λ, ρ) has the form

$$M(\sigma,\lambda,\rho) = \begin{bmatrix} M_1(\sigma,\lambda,\rho) & 0 \\ N(\sigma,\lambda,\rho) & M_2(\sigma,\lambda,\rho) \end{bmatrix}$$

where M_1 and M_2 are respectively the matrices which act on u and v as follows:

$$u \mapsto \begin{cases} \lambda u \rho^T \\ \lambda u^T \rho^T \end{cases} \qquad v \mapsto \begin{cases} \rho v \rho^T \\ \rho v^T \rho^T \end{cases}$$

and N the matrix such that $(u,0) \mapsto (0, \rho[u, \beta^T]\rho^T)$ (with some misuse of notation).

As tr M = tr M_1 + tr M_2, the character of the representation given by (2.62) is the same as that defined by (2.53). Since $\Sigma_{(Q,\beta)}$ is compact this means that the representations are isomorphic.

□

2.6.2 Isotropy subgroup slice representations

The adjoint representation of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ is defined by

$$(\sigma,\lambda,\rho)\cdot(\xi,\eta) = \begin{cases} (\lambda\xi\lambda^T, \rho\eta\rho^T) & \text{for} \quad \sigma = 1 \\ (\lambda\eta\lambda^T, \rho\xi\rho^T) & \text{for} \quad \sigma = \tau. \end{cases}$$

With respect to a suitable basis in $so(3) \times so(3)$ this can be written as

$$(\sigma,\lambda,\rho)\cdot(\xi,\eta) = \begin{cases} \begin{bmatrix} M_\lambda & 0 \\ 0 & M_\rho \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} & \text{for} \quad \sigma = 1 \\ \begin{bmatrix} 0 & M_\lambda \\ M_\rho & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} & \text{for} \quad \sigma = \tau. \end{cases}$$

Restricted to $\mathbf{Z}_2$ this gives the representation $\mathbf{Z}_2 = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\}$. Another equivalent representation of $\mathbf{Z}_2$ is given by

$$\mathbf{Z}_2 = \{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$$

The change of coordinates transforming the first $\mathbf{Z}_2$ representation into the second is:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The same change of coordinates transforms $\left\{\begin{bmatrix} M_\lambda & 0 \\ 0 & M_\rho \end{bmatrix}, \begin{bmatrix} 0 & M_\lambda \\ M_\rho & 0 \end{bmatrix}\right\}$ as follows:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} M_\lambda & 0 \\ 0 & M_\rho \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} M_\lambda + M_\rho & M_\lambda - M_\rho \\ M_\lambda - M_\rho & M_\lambda + M_\rho \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & M_\lambda \\ M_\rho & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} M_\lambda + M_\rho & -M_\lambda + M_\rho \\ M_\lambda - M_\rho & -M_\lambda - M_\rho \end{bmatrix}.$$

For any isotropy subgroup $\Sigma_{(Q,\beta)}$ we have $\lambda = \rho$. Thus the adjoint $\Sigma_{(Q,\beta)}$ action is given by:

$$(\sigma, \lambda, \lambda)\cdot(\xi,\eta) = \begin{cases} \begin{bmatrix} M_\lambda & 0 \\ 0 & M_\lambda \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} & \text{for } \sigma = 1 \\ \begin{bmatrix} M_\lambda & 0 \\ 0 & -M_\lambda \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} & \text{for } \sigma = \tau. \end{cases}$$

The $\Sigma_{(Q,\beta)}$-action on $gl(3)\times gl(3)^*$ is given in proposition 21 by (2.53). This action is just

$$(\sigma, \lambda, \rho)\cdot(u,v) = \begin{cases} (\lambda u \lambda^T, \lambda v \lambda^T) & \text{for } \sigma = 1 \\ (\lambda u^T \lambda^T, \lambda v^T \lambda^T) & \text{for } \sigma = \tau. \end{cases} \tag{2.63}$$

Decompose $gl(3)$ into $SO(3)$ irreducible spaces, $V_0 \oplus V_1 \oplus V_2$, where V_l is the spherical harmonics of order l. For $j = 0, 1$ and 2 these can be identified with the spaces of scalar multiples of the identity, skew-symmetric and symmetric trace zero matrices, respectively. The adjoint action of $\Sigma_{(Q,\beta)}$ on $V_0 \times V_0$ and $V_2 \times V_2$ is given (with respect to a suitable basis), by

$$(u,v) \mapsto \begin{bmatrix} M_\lambda & 0 \\ 0 & M_\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \quad \text{for } \sigma = 1 \text{ and } \tau$$

and on $V_1 \times V_1$ by

$$(u,v) \mapsto \begin{cases} \begin{bmatrix} M_\lambda & 0 \\ 0 & M_\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} & \text{for } \sigma = 1 \\ \begin{bmatrix} -M_\lambda & 0 \\ 0 & -M_\lambda \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} & \text{for } \sigma = \tau. \end{cases}$$

Let χ_l be the character of the representation of $SO(3)$ on V_l, χ_l^+ the character of the $\mathbf{Z}_2^\tau \times_s SO(3)^D$ representation on V_l which has τ acting trivially and χ_l^- the character with τ acting non-trivially.

Thus we have the the following result:

Proposition 23 *The character of the $\Sigma_{(Q,\beta)}$ representation on the $(V_1 \times V_1)$ factor of the tangent space is the restriction of $\chi_l^- + \chi_l^-$. The character of the adjoint representation of $\Sigma_{(Q,\beta)}$ on $(V_1 \times V_1)$ is the restriction of $\chi_l^+ + \chi_l^-$.*

Let us compute the slice representations for all the different isotropy subgroups of the isotropy lattice of the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$-action on the phase space. In order to do this we will use the character theory and formula (2.52).

First of all recall that, from lemma 8, if $(Q,\beta) \in GL^+(3) \times gl(3)^*$ and Q diagonal, then the isotropy subgroup $\Sigma_{(Q,\beta)}$ fixes (Q,β) if and only if it is of the form $(Q,\beta) = (Q, Q^{1/2}\alpha Q^{-1/2})$ (see table 2.4). Thus, from the expression of the momentum map in body coordinates, we have

$$J(Q,\beta) = (Q^{-1/2}\alpha Q^{1/2} - Q^{1/2}\alpha^T Q^{-1/2}, Q^{-1/2}\alpha^T Q^{1/2} - Q^{1/2}\alpha Q^{-1/2}).$$

If $\alpha = \alpha^T$ we have $J_1 = J_2$, and if $\alpha = -\alpha^T$, $J_1 = -J_2$. In such cases we have that the isotropy subgroup of $\mu = J(Q,\beta)$ is $\Sigma_\mu = \mathbf{Z}_2 \times (SO(2) \times SO(2))$.

When α is diagonal then $\Sigma_\mu = \mathbf{Z}_2 \times (SO(3) \times SO(3))$.

We will compute the slice representation for each isotropy subgroup by formula (2.52), that is by computing the characters $\chi(T_p\mathcal{P})$, $\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C})$ and $\chi(\frac{\gamma}{\gamma_\mu})$.

a) Slice representation of $\mathbf{Z}_2^\tau \times_s (SO(3))^D$.

In this case we have $\Sigma_\mu = \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$. Thus $\gamma = \gamma_\mu$. The respective characters are:

$$\chi(T_p\mathcal{P}) = 2\left(\chi_0^+ + \chi_1^- + \chi_2^+\right).$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\left(\chi_1^+ + \chi_1^- - \chi_1^+\right) = 2\chi_1^-.$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = 0.$$

Thus formula (2.52), gives for the slice representation character:

$$\chi(Y) = 2\left(\chi_0^+ + \chi_2^+\right)$$

Counting the dimensions we conclude that the $\Sigma_{(Q,\beta)}$ representation on the slice Y is 12 dimensional and isomorphic to the set of symmetric matrices.

b) Slice representation of $\mathbf{Z}_2^\tau \times_s (O(2))^D$

As in the last case we have $\Sigma_\mu = \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$.

Let χ_+ denote the trivial 1-dimensional representation of $O(2)$, χ_- the non-trivial 1-dimensional representation and χ_j ($j = 1, 2, ...$) the character of the 2-dimensional representations with kernel $\mathbf{Z}_j$, the cyclic group of order j.

Superscript + and − are used to denote the trivial and non-trivial action of τ. Then:

$$\chi(T_p\mathcal{P}) = 2\left(\chi_+^+ + \chi_-^- + \chi_1^- + \chi_+^+ + \chi_1^+ + \chi_2^+\right).$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\left(\chi_-^+ + \chi_1^+ + \chi_-^- + \chi_1^- - \chi_-^+\right) = 2\left(\chi_1^+ + \chi_-^- + \chi_1^-\right).$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = 0.$$

Thus

$$\chi(Y) = 4\,\chi_+^+ + 2\,\chi_2^+$$

This is a 8-dimensional slice representation.

c) Slice representation of $\mathbf{Z}_2^\tau \times_s (D_2)^D$

As before $\Sigma_\mu = \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$. On the other hand we have $\gamma_p = \{0\}$.

Take D_2 to be generated by $\rho_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\rho_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Let $\chi_{(i,j)}$ be the character of the 1-dimensional representation with ρ_1 acting trivially if $i = 0$ and non-trivially if $i = 1$ and ρ_2 acting trivially if $j = 0$ and non-trivially if $j = 1$. As above we use a superscript $+$ or $-$ if τ acts trivially or non-trivially, respectively. Then:

$$\chi(T_p\mathcal{P}) = 2\left(\chi_{(0,0)}^+ + \chi_{(0,1)}^- + \chi_{(1,0)}^- + \chi_{(1,1)}^- + 2\chi_{(0,0)}^+ + \chi_{(0,1)}^+ + \chi_{(1,0)}^+ + \chi_{(1,1)}^+\right).$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\left(\chi_{(0,1)}^+ + \chi_{(1,0)}^+ + \chi_{(1,1)}^+ + \chi_{(0,1)}^- + \chi_{(1,0)}^- + \chi_{(1,1)}^-\right).$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = 0.$$

So

$$\chi(Y) = 6\,\chi_{(0,0)}^+.$$

This is the 6-dimensional trivial representation.

d) Slice representation of $(\widetilde{O(2)})^D$

We use the same notations for characters of $\Sigma_p \cong O(2)$ as in **b**). In this case we have $\Sigma_\mu = \mathbf{Z}_2^\tau \times_s (SO(2) \times SO(2))$. The characters of $\mathbf{Z}_2^\tau \times O(2)^D$ restrict to $(\widetilde{O(2)})^D$ as follows:

$$\chi_\pm^+ \to \chi_+ \qquad \chi_\pm^- \to \chi_\mp \qquad \chi_j^\pm \to \chi_j$$

$$\chi(T_p\mathcal{P}) = 2\,(\chi_+ + \chi_+ + \chi_1 + \chi_+ + \chi_1 + \chi_2).$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\,(\chi_+ + \chi_- - \chi_-) = 2\chi_+.$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = \chi_- + 2\,\chi_1 + \chi_+ - \chi_+ - \chi_- = 2\,\chi_1.$$

Then

$$\chi(Y) = 4\,\chi_+ + 2\,\chi_1 + 2\,\chi_2.$$

e) Slice representation of $(\mathbf{Z}_2^\tau \times \mathbf{Z}_2)^D$

The isotropy subgroup Σ_μ is $\mathbf{Z}_2^\tau \times (SO(2) \times SO(2))$. Let χ_+ and χ_- denote the trivial and non-trivial character of $\mathbf{Z}_2$ respectively. Use the superscript $\pm$ to denote the action of τ, as in previous cases.

$$\chi(T_p\mathcal{P}) = 2\left(\chi_+^+ + \chi_+^- + 2\,\chi_-^- + 3\,\chi_+^+ + 2\chi_-^+\right).$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\left(\chi_+^+ + \chi_+^-\right).$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = \left(\chi_+^+ + \chi_+^- + 2\,\chi_-^+ + 2\,\chi_-^- - \chi_+^+ - \chi_+^-\right) = 2\left(\chi_-^+ + \chi_-^-\right).$$

Then

$$\chi(Y) = 6\chi_+^+ + 2\chi_-^+ + 2\chi_-^-$$

This is a 10-dimensional representation.

f) Slice representation of $\widetilde{D_2}^D$

The isotropy subgroup Σ_μ is in this case $\mathbf{Z}_2^\tau \times_s (SO(2) \times SO(2))$ and the Lie algebra γ_p is $\{0\}$.

Take $\widetilde{D_2}^D$ to be generated by ρ_1 and $\tau \circ \rho_2$ where ρ_1 and ρ_2 are as in (c). Use the same notation for characters of $\widetilde{D_2}^D \cong D_2^D$ as described in (c). Then the characters of $\mathbf{Z}_2 \times_s D_2^D$ restrict to $\widetilde{D_2}^D$ as follows:

$$\chi^+_{(i,j)} \rightarrow \chi_{(i,j)}; \qquad \chi^-_{(i,0)} \rightarrow \chi_{(i,1)}; \qquad \chi^-_{(i,1)} \rightarrow \chi_{(i,0)}.$$

Hence

$$\chi(T_p\mathcal{P}) = 2\left(\chi_{(0,0)} + \chi_{(0,0)} + \chi_{(1,1)} + \chi_{(1,0)} + 2\chi_{(0,0)} + \chi_{(0,1)} + \chi_{(1,0)} + \chi_{(1,1)}\right).$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\left(\chi_{(0,0)} + \chi_{(0,1)}\right).$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = \chi_{(1,1)} + \chi_{(0,0)} + \chi_{(1,0)} + \chi_{(1,0)} + \chi_{(0,1)} + \chi_{(1,1)} - \chi_{(0,0)} - \chi_{(0,1)}$$

$$= 2\left(\chi_{(1,0)} + \chi_{(1,1)}\right).$$

So

$$\chi(Y) = 6\,\chi_{(0,0)} + 2\,\chi_{(1,0)} + 2\,\chi_{(1,1)}$$

and we get a 10-dimensional representation.

g) Slice representation of $\mathbf{Z}_2^D$

In this case Σ_μ can be either:

1. $\Sigma_\mu = \mathbf{1} \times_s (SO(3) \times SO(2))$.
2. $\Sigma_\mu = \mathbf{1} \times_s (SO(2) \times SO(2))$.

As in (e) χ_+ and χ_- denote the trivial and non-trivial character of $\mathbf{Z}_2$.

Case 1: $\Sigma_\mu = \mathbf{1} \times_s (SO(3) \times SO(2))$

$$\chi(T_p\mathcal{P}) = 2\,(\chi_+ + \chi_+ + 2\,\chi_- + 3\,\chi_+ + 2\chi_-) = 10\,\chi_+ + 8\,\chi_-.$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\,(2\,\chi_+ + 2\,\chi_-).$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = 2\,\chi_+ + 4\,\chi_- - 2\,\chi_+ - 2\,\chi_- = 2\,\chi_-.$$

Then

$$\chi(Y) = 6\chi_+ + 2\chi_-$$

which is a 8-dimensional representation.

Case 2: $\Sigma_\mu = \mathbf{1} \times_s (SO(2) \times SO(2))$

$$\chi(T_p\mathcal{P}) = 2\,(\chi_+ + \chi_+ + 2\,\chi_- + 3\,\chi_+ + 2\chi_-) = 10\,\chi_+ + 8\,\chi_- \qquad \text{as in (1.).}$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 4\,\chi_+.$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = 2\,\chi_+ + 4\,\chi_- - 2\,\chi_+ = 4\,\chi_-.$$

Then

$$\chi(Y) = 6\chi_+ + 4\chi_-$$

which is a 10-dimensional representation.

h) Slice representation of $\mathbf{Z}_2^\tau$

We have $\Sigma_\mu = \mathbf{Z}_2^\tau \times_s (SO(2) \times SO(2))$. We use χ^+ and χ^- to denote the trivial and non-trivial representation of $\mathbf{Z}_2^\tau$, respectively.

$$\chi(T_p\mathcal{P}) = 12\,\chi^+ + 6\,\chi^-.$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\,\chi^+ + 2\,\chi^-.$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = 3\,\chi^+ + 3\,\chi^- - \chi^+ - \chi^- = 2\,\chi^+ + 2\,\chi^-.$$

Then

$$\chi(Y) = 8\,\chi^+ + 2\,\chi^-.$$

That is a 10-dimensional representation.

i) Slice representation of $\widetilde{\mathbf{Z}_2}^D$

$\Sigma_\mu = \mathbf{Z}_2^\tau \times_s (SO(2) \times SO(2))$. We use $\chi_\pm$ to denote the characters of $\widetilde{\mathbf{Z}}_2^D \cong \mathbf{Z}_2$. The character of $\mathbf{Z}_2^\tau \times \mathbf{Z}_2^D$ restrict as follows:

$$\chi_\pm^+ \to \chi_\pm; \qquad \chi_\pm^- \to \chi_\mp.$$

Then

$$\chi(T_p\mathcal{P}) = 12\,\chi_+ + 6\,\chi_-,$$

$$\chi(\frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C}) = 2\,\chi_+ + 2\,\chi_-,$$

$$\chi(\frac{\gamma}{\gamma_\mu}) = 2\,\chi_+ + 2\,\chi_-.$$

Thus

$$\chi(Y) = 8\,\chi_+ + 2\,\chi_-.$$

That is a 10-dimensional representation.

j) Slice representation of 1

Again there are two cases to consider:

1. $\Sigma_\mu = \mathbf{1} \times_s (SO(3) \times SO(2))$.
2. $\Sigma_\mu = \mathbf{1} \times_s (SO(2) \times SO(2))$.

The representation of Σ_p are trivial, so only the dimension is relevant.

1. $\dim(T_p\mathcal{P}) = 18$.
   $\dim \frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C} = 8$.
   $\dim \frac{\gamma}{\gamma_\mu} = 2$.
   Thus
   $$\dim Y = 8.$$
2. $\dim(T_p\mathcal{P}) = 18$.
   $\dim \frac{\gamma_\mu}{\gamma_p} \otimes \mathbf{C} = 4$.
   $\dim \frac{\gamma}{\gamma_\mu} = 4$.
   Thus
   $$\dim Y = 10.$$

We summarize these results in the table 2.6.

Σ_p	Σ_μ	$\chi(Y)$	$\dim Y$
$\mathbf{Z}_2^\tau \times SO(3)^D$	$\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$	$2\left(\chi_0^+ + \chi_2^+\right)$	12
$\mathbf{Z}_2^\tau \times O(2)^D$	$\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$	$4\,\chi_+^+ + 2\,\chi_2^+$	8
$\mathbf{Z}_2^\tau \times D_2^D$	$\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$	$6\,\chi_{(0,0)}^+$	6
$\widetilde{O(2)}^D$	$\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$	$4\,\chi_+ + 2\,\chi_1 + 2\,\chi_2$	12
$\mathbf{Z}_2^\tau \times \mathbf{Z}_2^D$	$\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$	$6\,\chi_+^+ + 2\,\chi_-^+ + 2\,\chi_-^-$	10
$\widetilde{D_2}^D$	$\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$	$6\,\chi_{(0,0)} + 2\,\chi_{(1,0)} + 2\,\chi_{(1,1)}$	10
$\mathbf{Z}_2^D$	$\mathbf{1} \times_s (SO(3) \times SO(2))$	$6\,\chi_+ + 2\,\chi_-$	8
	$\mathbf{1} \times_s (SO(2) \times SO(2))$	$6\,\chi_+ + 4\,\chi_-$	10
$\mathbf{Z}_2^\tau$	$\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$	$8\,\chi^+ + 2\,\chi^-$	10
$\widetilde{\mathbf{Z}_2}^D$	$\mathbf{Z}_2 \times_s (SO(2) \times SO(2))$	$8\,\chi_+ + 2\,\chi_-$	10
$\mathbf{1}$	$\mathbf{1} \times_s (SO(3) \times SO(2))$		8
	$\mathbf{1} \times_s (SO(2) \times SO(2))$		10

Table 2.6: Symplectic slice representations of the isotropy subgroups of the action of $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ on $T^*GL^+(3)$. The characters are defined in the text.

2.7 Slice Reduction for the Affine Rigid Body

The first chapter was devoted to the study of reduction of symmetric Hamiltonian systems, describing different reduction procedures, particularly for singular points of the momentum map. Here we will apply the slice reduction treated in section 1.5 to the affine rigid body. This reduction works in a neighbourhood of an isotropically embedded group orbit $\mathcal{O}_p$, through a point p of the phase space (which may or not be a singular point of the momentum map). The hypothesis of $\mathcal{O}_p$ being isotropically embedded is not a very restrictive one since, as mentioned on page 25, we can always reduce the study of a nonzero level set of a momentum map to the study of the zero level set of another momentum map by the so-called shifting trick. For points p belonging to a zero level set of an equivariant momentum map the group orbit through p, $\mathcal{O}_p$, is an isotropic embedding.

Slice reduction is given by theorem 3 and is closely related to the application of the normal form for the momentum map. It gives, in a neighbourhood of a group orbit through p, the so-called MMW-reduced space $J^{-1}(\mu)/G_\mu$ (for μ in a neighbourhood U of zero in the dual of the Lie algebra $\mathcal{G}$) isomorphic as a Poisson variety to

$$\frac{\Psi^{-1}(0) \cap (G \cdot \mu \times U)}{G_p}$$

where $\Psi : \mathcal{G}^* \times Y \to \mathcal{G}_p^*$ is an equivariant map for which the restriction to the product of the adjoint group orbit through μ, $G \cdot \mu = \mathcal{O}_\mu$, by the symplectic slice Y, is the momentum map for the action of the isotropy subgroup of p, G_p, on $\mathcal{G}^* \times Y$.

The proof of theorem 3 gives $J^{-1}(\mu)/G_\mu \cong J^{-1}(\mathcal{O}_\mu)/G$ isomorphic to $\phi^{-1}(\mathcal{O}_\mu)/G_p$ where $\phi^{-1}(\mathcal{O}_\mu)$ is the set of points $(\nu, y) \in (\mathcal{G}/\mathcal{G}_p)^* \times Y$ such that $(\nu + J_Y(y)) = \mathcal{O}_\mu$. Here J_Y is the momentum map for the G_p-action on the symplectic slice Y and $(\mathcal{G}/\mathcal{G}_p)^*$ is identified with

$$(\mathcal{G}/\mathcal{G}_p)^* = \{\alpha \in \mathcal{G}^* \ : \ \langle \alpha, \xi \rangle = 0 \quad \text{for all} \quad \xi \in \mathcal{G}_p\}. \tag{2.64}$$

The slice Y defined as $Y = Ker\ dJ_p/W$, where $W = Ker\ dJ_p \cap T_p(G \cdot p)$, is symplectic with a G_p invariant symplectic form ω_Y induced from the symplectic form ω on $\mathcal{P}$, (see Montaldi *et al.* [38]). For the left identification of $\mathcal{P} = T^*GL^+(3)$ with $GL^+(3) \times gl(3)^*$ the symplectic form ω, at $(Q, \beta) \in GL^+(3) \times gl(3)^*$, is given by

$$\omega_{(Q,\beta)}((Qu_1, \rho_1), (Qu_2, \rho_2)) = \langle \rho_2, u_1 \rangle - \langle \rho_1, u_2 \rangle + \langle \beta, [u_1, u_2] \rangle$$

where $u_i \in T_e GL^+(3) = gl(3)$ and $\rho_i \in gl(3)^*$ (for $i = 1, 2$) (see Abraham and Marsden [1] pg. 315, or chapter 1 pg. 33). Furthermore given a symplectic representation Y of G_p this induces a representation of $\mathcal{G}_p$ on Y for which $\xi \mapsto A_\xi \in \mathcal{SP}(Y)$, where $\mathcal{SP}$ denotes the group of symplectic matrices. The momentum map J_Y is a homogeneous quadratic form given by

$$\langle J_Y(y), \xi \rangle = \frac{1}{2} \omega_Y(A_\xi y, y).$$

Let us see how theorem 3 works for the case $G_p = \mathbf{Z}_2 \times_s SO(3)^D$. In this case $J(Q,\beta) = (0,0)$ and so the group orbit through p is isotropically embedded in $\mathcal{P}$ allowing the application of the normal form theorem.

Let $(S,V) \in (so(3)^* \times so(3)^*)$ be a point sufficiently close to $(0,0) \in (so(3)^* \times so(3)^*)$. We would like to obtain the reduced space

$$M_{(S,V)} = \frac{J^{-1}(S,V)}{G_{(S,V)}}$$

where J is the momentum map for the $SO(3) \times SO(3)$- action on $\mathcal{P} = T^*GL^+(3)$ and $G_{(S,V)}$ the isotropy subgroup of (S,V) with respect to the coadjoint representation.

The normal form for the momentum map is obtained by considering suitable $G = \mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ and G_p actions on $T^*(\mathbf{Z}_2 \times_s SO(3) \times SO(3))^- \times Y$ where Y is the symplectic slice. When $G_p = \mathbf{Z}_2 \times_s SO(3)^D$ the symplectic slice is isomorphic to the set $SY \times SY \subset (gl(3) \times gl(3)^*)$ where SY is the set of symmetric matrices.

We use the left identification of $T^*(\mathbf{Z}_2 \times_s (SO(3) \times SO(3)))$ with

$$(\mathbf{Z}_2 \times_s SO(3) \times SO(3)) \times (so(3)^* \times so(3)^*) .$$

Let $(\sigma, K, T;\ \mu_L, \mu_R; y_1, y_2)$ be an element of $(\mathbf{Z}_2 \times_s SO(3) \times SO(3)) \times (so(3)^* \times so(3)^*)$. The G and G_p actions needed for the normal form are the following ones.

i) The G action is:

$$((\sigma_1, K_1, T_1), (\sigma, K, T;\ \mu_L, \mu_R; y_1, y_2)) \mapsto (\sigma_1\sigma, (K_1, T_1)(\sigma_1 \cdot (K,T)); \mu_L, \mu_R; y_1, y_2) .$$

ii) Any $(\sigma_1, \lambda, \lambda) \in G_p = \mathbf{Z}_2 \times_s SO(3)^D$ acts by:

$$\begin{aligned} &((\sigma_1, \lambda, \lambda), (\sigma, K, T;\ \mu_L, \mu_R; y_1, y_2)) \\ &\quad = \left(\sigma\sigma_1, K\lambda^T, T\lambda^T; \overline{Ad}^*_{(\lambda,\lambda)}(\sigma \cdot (\mu_L, \mu_R)); \Phi_{(\lambda,\lambda)}(y_1, y_2)\right) . \end{aligned}$$

Note that in the above definition we have used the fact that $\sigma \cdot (\lambda, \lambda) = (\lambda, \lambda)$ for any $\sigma \in \mathbf{Z}_2$. Φ denotes the G_p action on Y.

The momentum maps J_G and J_{G_p} for the actions defined in **i)** and **ii)** on $T^*G^- \times Y$ are given by

$$\begin{aligned} J_G(\sigma, K, T, \mu_L, \mu_R; y_1, y_2) &= \overline{Ad}^*_{(K^T,T^T)}(\sigma \cdot (\mu_L, \mu_R)) \\ &= \begin{cases} (K\mu_L K^T, T\mu_R T^T) & \text{for} \quad \sigma = 1 \\ (K\mu_R K^T, T\mu_L T^T) & \text{for} \quad \sigma = \tau \end{cases} \end{aligned}$$

$$J_{G_p}(\sigma, K, T;\ \mu_L, \mu_R;\ y_1, y_2) = i^*(\sigma, \mu_L, \mu_R) + J_Y(y_1, y_2)$$

where J_Y is the momentum map for the G_p-action on Y and $i^* : \mathcal{G}^* \to \mathcal{G}_p^*$ is the dual of the projection $i : \mathcal{G}_p \to \mathcal{G}$.

With $(\mathcal{G}/\mathcal{G}_p)^*$ identified with the $Kernel$ of i^*, the set $J_{G_p}^{-1}(0,0)$ can be parametrized by $G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y$ and the level set $J^{-1}(\mathcal{O}_{(S,V)})$ is then the set of equivalence classes

$$[K, T, \nu_L, \nu_R; y_1, y_2] \in \frac{G \times (\mathcal{G}/\mathcal{G}_p)^* \times Y}{G_p}$$

for which

$$(\nu_L, -\nu_L) + J_Y(y_1, y_2) = \overline{Ad}^*_{(K,T)}(\sigma \cdot (S, V))$$

since $(\nu_L, \nu_R) \in ker\, i^*$ implies $\nu_L = -\nu_R$.

Theorem 4 states that we can (partially) reduce the dynamics generated by a $\mathbf{Z}_2 \times_s SO(3) \times SO(3)$ invariant Hamiltonian near the group orbit $\mathcal{O}_{(Q,\beta)}$ on $\mathcal{P}$ to the dynamics of a $\mathbf{Z}_2 \times_s SO(3)^D$ invariant function $\hat{H} : \mathcal{G}^* \times Y \to \mathbf{R}$. In particular the flow of H on $M_{(S,V)}$ is mapped to the flow of the restriction, $\hat{H}_{(S,V)}$, of $\hat{H}$ to $\mathcal{O}_{(S,V)} \times Y$.

As an example we apply this reduction to the dynamics near spherical equilibria. We can distinguish several cases for (S, V) for which we will get different reduced dynamics.

Proposition 24 *Let $(Q, \beta) \in GL^+(3) \times gl(3)^* = \mathcal{P}$ be such that its isotropy subgroup for the $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$-action is $G_p = \mathbf{Z}_2 \times_s SO(3)^D$. Let (S, V) be sufficiently close to $(0,0) \in so(3)^* \times so(3)^*$. Then the dynamics generated by a $\mathbf{Z}_2 \times_s (SO(3) \times SO(3))$ invariant Hamiltonian function H on $\mathcal{P}$ for which (Q, β) is an equilibrium point reduces, near (Q, β), to*

1. *$SO(3)^D$ equivariant dynamics on $J_Y^{-1}(0) \subset Y$ for $S = V = 0$.*
2. *$SO(3)^D$ equivariant dynamics on $S_s^2 \times Y$ with $J_Y(y) = i^*(S, V)$, for $S \neq 0$ and $V = 0$, where S_s^2 denotes the sphere in $\mathbf{R}^3$ of radius $\|s\|$ where $s \in \mathbf{R}^3$ is isomorphic to $S \in so(3)$.*
3. *$\mathbf{Z}_2 \times_s SO(3)^D$ equivariant dynamics on $S_s^2 \times S_v^2 \times Y$ with $J_Y(y) = i^*(S, V)$, for $S = V \neq 0$ where S_s^2, S_v^2 are the spheres defined above.*
4. *$SO(3)^D$ equivariant dynamics on $S_s^2 \times S_v^2 \times Y$ with $J_Y(y) = i^*(S, V)$, for $S \neq V \neq 0$ where S_s^2, S_v^2 are the spheres defined above.*

Proof : First of all note that the $SO(3)$ coadjoint group orbit through a point $S \in so(3)^*$ is a 2-sphere of radius $\|s\|$ where s is the $\mathbf{R}^3$ vector identification of S.

For 1), the remark of page 43 shows that the dynamics reduces to a $\mathbf{Z}_2 \times_s SO(3)^D$ equivariant dynamics on $J_Y^{-1}(0,0)$. As Y is isomorphic to the set $SY \times SY$ on which τ acts trivially, the action reduces to one of $SO(3)^D$.

For all the other cases, theorem 4 says that the dynamics reduces to G_p equivariant dynamics on the product of the coadjoint group orbit throught (S, V) by Y. By definition of the action of $G_p = \mathbf{Z}_2 \times_s SO(3)^D$ on $so(3)^* \times so(3)^*$ this is given by

$$(\sigma, \lambda, \lambda) \cdot (S, V) = \begin{cases} (\lambda S \lambda^T, \lambda V \lambda^T) & \text{for} \quad \sigma = 1 \\ \\ (\lambda V \lambda^T, \lambda S \lambda^T) & \text{for} \quad \sigma = \tau \end{cases}$$

which means that in cases 2) and 4) the action reduces to one of $SO(3)^D$.

$\square$

2.7.1 Applications to bifurcation of relative equilibria

As we have mentioned, if p is a relative equilibrium on $\mathcal{P}$ and the group orbit $\mathcal{O}_p$ is an orbit of relative equilibria of the G-invariant Hamiltonian on $\mathcal{P}$, then $D_y\widehat{H}(0,0) = 0$. In proposition 9 it is shown that the equations of motion for the reduced dynamics, in appropriate coordinates on $\mathcal{G}^* \times Y$, are given by the following set of equations

$$\begin{aligned} \dot{\xi} &= \left[\xi, D_\xi\widehat{H}(\xi,y)\right] \\ \dot{y} &= SD_y\widehat{H}(\xi,y) \end{aligned}$$

where $y \in Y$ and $\xi \in \mathcal{G}$, and $\mathcal{G}$ has been identified with $\mathcal{G}^*$.

Given a subgroup Σ of G_p, these equations restrict to equations on the fixed point set $Fix(\Sigma;\mathcal{G}^* \times Y) = Fix(\Sigma;\mathcal{G}^*) \times Fix(\Sigma;Y)$. If the Lie bracket is trivial on $Fix(\Sigma;\mathcal{G})$, i.e $[\xi,\eta] = 0$ for all ξ,η in $Fix(\Sigma;\mathcal{G})$, then the restricted vector field on $Fix(\Sigma;\mathcal{G}^* \times Y)$ is trivial in the $Fix(\Sigma;\mathcal{G}^*)$ direction and the restricted system is a family of Hamiltonian systems on $Fix(\Sigma;Y)$ parametrized by $Fix(\Sigma;\mathcal{G}^*)$.

Defining a nondegenerate relative equilibrium orbit $\mathcal{O}_p$ by the condition that $D^2_{yy}\widehat{H}(0,0)$ has maximal rank, an application of the implicit function theorem gives the following result.

Theorem 7 *Let $\mathcal{O}_p$ be a nondegenerate relative equilibrium of the G-invariant Hamiltonian system. Suppose the isotropy subgroup G_p is finite and $J(\mathcal{O}_p) = 0$. Let Σ be a subgroup of G_p such that the Lie bracket is trivial on $Fix(\Sigma;\mathcal{G}^*)$. Then there exists a neighbourhood U of zero in $Fix(\Sigma;\mathcal{G}^*)$ such that for all μ in U there is a relative equilibrium $\mathcal{O}_\mu = G\cdot p_\mu$ near $\mathcal{O}_p$ with $J(\mathcal{O}_\mu) = \mu$ and isotropy subgroup G_{p_μ} containing Σ.*

Note that if $dim\ Fix(\Sigma;\mathcal{G}^*) = 1$ the Lie bracket is always trivial and $\{\xi\} \times Fix(\Sigma;Y)$ is invariant for all $\xi \in Fix(\Sigma;\mathcal{G})$ with dynamics given by $\widehat{H}(\cdot,\xi)\big|_{Fix(\Sigma,Y)} = \widehat{H}_{\Sigma,\xi}$.

Let us apply the above theorem to the existence of S-type ellipsoids of equilibrium for the problem due to Dirichlet in a neighbourhood of the group orbit $\mathcal{O}_p$.

Let $(Q,\beta) = p$ be a relative equilibrium such that its isotropy subgroup is $G_p = \mathbf{Z}_2 \times_s D_2^D$. The Lie algebra $\mathcal{G}_p$ only contains the zero element and $J(\mathcal{O}_p) = (0,0)$. Theorem 3 gives the reduced dynamics in a neighbourhood of $\mathcal{O}_p$. Consider (S,V) to be sufficiently close to $(0,0)$ in $\mathcal{G}^*$ and $p_{(S,V)} \in \mathcal{P}$ such that $J(p_{(S,V)}) = (S,V)$.

Proposition 25 *Let p be a nondegenerate ellipsoidal equilibrium.*

Then for each (S,V) sufficiently close to $(0,0)$, and such that both S and V are parallel to the same principal axis of p, there exists an 'S-type' relative equilibrium $p_{(S,V)}$, near p with $J(p_{(S,V)}) = (S,V)$. If *i)* $S = 0$ *then* $p_{(S,V)}$ *has angular momentum equal 0.*

ii) $V = 0$ *then* $p_{(S,V)}$ *has zero circulation.*

iii) $S = V$ *then* $p_{(S,V)}$ *is self-adjoint.*

iv) $S = -V$ *then* $p_{(S,V)}$ *is symmetric-adjoint.*

Proof: The isotropy subgroup of p is $\mathbf{Z}_2^\tau \times_s D_2^D$ (see table 2.5). The symplectic slice Y is 6-dimensional and has a trivial action of $\mathbf{Z}_2^\tau \times_s D_2^D$ (see table 2.6). There are 3 cases to consider:

1. $S = 0$ or $V = 0$.

   The dynamics reduce to a D_2^D equivariant system on $S^2 \times Y$. Each subgroup $\mathbf{Z}_2$ in D_2^D has fixed point set in S^2 consisting of 2 points. Theorem 7 yields 2 relative equilibria for each of the 3 subgroups. These are 'S-type' ellipsoids rotating about the principal axis of p with zero circulation or zero angular momentum.

2. $\|S\| = \|V\| \neq 0$.

   The dynamics reduce to a $\mathbf{Z}_2^\tau \times_s D_2^D$ system on $S^2 \times S^2 \times Y$. The isotropy subgroups of the action of $\mathbf{Z}_2^\tau \times_s D_2^D$ on $S^2 \times S^2$ with 0-dimensional fixed point sets are $\mathbf{Z}_2^\tau \times_s \mathbf{Z}_2^D$ and $\widetilde{D_2}^D$. These yield 'S-type' relative equilibria which are self-adjoint and symmetric adjoint, respectively.

3. $\|S\| \neq \|V\|$, $S \neq 0$, $V \neq 0$.

   The dynamics reduce to a D_2^D system on $S^2 \times S^2 \times Y$. Each subgroup $\mathbf{Z}_2$ in D_2^D has a fixed point set in $S^2 \times S^2$ consisting of 4 points. Each of these gives a general 'S-type' relative equilibrium rotating about the principal axis corresponding to that of $\mathbf{Z}_2$.

□

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